A NEW APPROACH WITH QUASI-DUAL RECIPROCITY BOUNDARY ELEMENT METHOD APPLIED TO DIFFUSIVE-ADVECTIVE EQUATION FOR POTENTIAL FLUID FLOW

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Abstract. *In this work an alternative boundary formulation to convective heat transfer for potential fluid flow is presented. The recent Quasi-Dual approach is used, but the main purpose is to rewrite the governing integral equation in terms of the potential velocity function and to divide the convective integral in four terms. Three of them are integrated without approximations and the last one is transformed according to the Quasi-Dual formulation procedure. The aim is to achieve better results than with other formulations. Using constant boundary elements, the numerical solution of one and two-dimensional problems is shown and these results are compared with analytical ones, in order to prove the good performance of the proposed formulation.*

Keywords. *Diffusive-Advective Equation, Heat Transfer Problems, Boundary Element Applications, Numerical Methods.*

1. Introduction

The Dual Reciprocity formulation is one of the most important techniques that the Boundary Element Method (BEM) has to amplify its application field and thus to reach a high level of acceptation among the engineering community, such as the Finite Element Method. Proposed by Nardini and Brebbia (1982), the main idea of the Dual Reciprocity is to employ auxiliary interpolation functions to establish easily the boundary integral formulation for many engineering problems, which do not have a suitable mathematical model to be analytically manipulated.

There are many well established analytical procedures to work out integral sentences, but still the associated boundary formulation is not simple for many applied mechanics fields, such as: fluid flow, heat transfer with convection, and other problems related to mass and energy transport. The complexity and the difficulty to elaborate fundamental solutions discourage engineers and other professionals to use the BEM as a tool to solve a number of daily industrial problems. Dual Reciprocity makes the method more simple and general. It also makes the BEM more attractive to industry professionals dealing with numerical modeling.

In the beginning of the nineties, Partridge, Brebbia and Wrobel (1992) presented a new and interesting generalization of the Dual Reciprocity formulation. They applied it to advective problems, using the simple diffusive fundamental solution, following an approach where spatial derivatives of the first order terms played the role of “internal domain source terms". In the present paper, source term is a generic denomination to any term operated by the Dual Reciprocity procedure, which is approximated by special interpolation functions. Unfortunately, the Dual Reciprocity approach is only efficient when the diffusive phenomenon is more significant than the convective one, that is, the results are good for small Peclet number.

Before the adaptation of Dual Reciprocity to modeling advective cases, there was another elegant and very accurate BEM formulation, which uses a fundamental solution related to a convective problem. Honma, Tanaka and Kaji (1985), Wrobel and De Figueiredo (1991) and Singh and Tanaka (1998) are some of the researchers who employed this “classical” BEM formulation to solve advective problems. However, it has a very serious limitation: the velocity field needs to be constant along all control volume. Using this classical procedure, BEM analysis of advective problems with space varying velocity was only possible after the adaptation of the Dual Reciprocity approach to deal with this problem according Ramachandran (1994).

Recently a new technique was presented by Loeffler and Mansur and Massaro and Loeffler (2001): the Quasi-Dual Reciprocity formulation. This formulation achieves very good results in convective heat transfer for potential fluid flow with median Peclet numbers, that is, when the convective phenomena is not absolute, independent of the velocity field being constant or variable. This formulation showed better accuracy and flexibility in comparison with former alternative boundary formulations, such as traditional Dual Reciprocity and the classical technique that uses diffusive-advective fundamental solution.

The good results of Quasi-Dual formulation in convective heat transfer stimulated the continuity of research about alternative formulations based on the same idea. In this work an adaptation of original Quasi-Dual formulation is presented, using a similar interpolation of derivatives employed in the traditional Dual Reciprocity model, but with the integral equation rewritten in terms of the potential velocity function and divided in three terms. Only one of them is approximated by an interpolation procedure. This formulation is named here Hybrid Quasi-Dual Reciprocity and for sake of simplicity it will be referred HQDR from now on.

One and two-dimensional examples are solved by HQRD and two of them are shown in this paper. The results are compared against those obtained from analytical and traditional Dual-Reciprocity formulation (TDR). The convergence of the method is evaluated through analyses where the mesh is successively refined for various Peclet numbers, in order to assess the effect of the approximation in the advective term.

2. The governing equation

Let a control volume Ω represent a region where stationary potential fluid flow occurs, and X represents a point with coordinates (x_1,x_2) in the two-dimensional space. In indicial notation, the governing equation of advective problems is given by:

$$K\theta_{,ii} - v_i\theta_{,i} = 0 \tag{1}$$

Where θ is the temperature, K is a scalar describing the thermal properties of a homogeneous continuum media and v_i denotes the components of the velocity vector of a particle. The essential and natural boundary conditions are defined by:

$$\theta = \bar{\theta} \qquad \text{at } \Gamma u \quad \text{(essential condition)} \tag{2}$$

$$K\theta_{,i}n_i - \theta v_i n_i = \bar{f} \qquad \text{at } \Gamma q \quad \text{(natural condition)} \tag{3}$$

In equations (2) and (3), $\Gamma = \Gamma u \cup \Gamma q$ represents the boundary of the control volume and n_i represents the components of the outward unity vector, normal to the boundary surface. Γu is the part of boundary where the essential condition is prescribed and Γq is the part where the complete natural condition f is known, including terms related to diffusive and advective flux.

It is interesting to express the governing equation in another form. Considering the incompressibility condition in the potential fluid flow, given by:

$$v_{i,i} = 0 \tag{4}$$

It is possible to rewrite the governing differential equation as shown below:

$$K\theta_{,ii} - (v_i\theta)_{,i} = 0 \tag{5}$$

The transformation of the original governing equation (1) into equation (5) is the basic requirement to establish the Quasi-Dual formulation. In this way it is possible to transform the convective term in other new three terms, where just one is approximated by the Dual Reciprocity interpolation procedure. The others receive the usual BEM treatment.

3. Hybrid Quasi- Dual Formulation

The formulation described here follows the traditional BEM approach, and the diffusive fundamental solution θ^* is employed to transform equation (5) in a strong integral form. When equation (5) is multiplied by θ^* and integrated over the volume, one has:

$$K\int_\Omega \theta_{,ii}\theta^* d\Omega = \int_\Omega (v_i\theta)_{,i}\theta^* d\Omega \tag{6}$$

According to Brebbia (1978), θ^* is given by:

$$\theta^* = (-1/2\pi)\ln r\,(\xi;X) \tag{7}$$

The mathematical treatment of the left hand side of equation (6) is trivial (see Brebbia (1978)); attention must be given to its right hand side. Using the integration by parts procedure, it can be written:

$$\int_\Omega (v_i\theta)_{,i}\theta^* d\Omega = \int_\Omega (v_i\theta\theta^*)_{,i} d\Omega - \int_\Omega v_i\theta\theta^*_{,i} d\Omega \tag{8}$$

Using the Divergence Theorem:

$$\int_\Omega (v_i\theta)_{,i}\theta^* d\Omega = \int_\Gamma v_i n_i \theta\theta^* d\Gamma - \int_\Omega v_i \theta\theta^*{}_{,i} d\Omega \tag{9}$$

Considering the condition of potential fluid flow, a potential of velocities Φ must exist such that:

$$v_i = \Phi_{,i} \tag{10}$$

Substituting equation (10) into expression (9) and using integration by parts, the following identity can be written:

$$\int_\Omega v_i \theta\theta^*{}_{,i} d\Omega = \int_\Omega \Phi_{,i}\theta\theta^*{}_{,i} d\Omega = \int_\Omega [\Phi\,\theta\theta^*{}_{,i}]_{,i} d\Omega - \int_\Omega \Phi\,[\theta\theta^*{}_{,i}]_{,i} d\Omega \tag{11}$$

Effecting the product derivative in the last term of the right hand side of equation (11) and applying the divergence theorem once more, it has:

$$\int_\Omega \Phi_{,i}\theta\theta^*{}_{,i} d\Omega = \int_\Gamma \Phi\,\theta q^* d\Gamma - \int_\Omega \Phi\,\theta\theta^*_{,ii} d\Omega - \int_\Omega \Phi\,\theta_{,i}\theta^*{}_{,i} d\Omega \tag{12}$$

In last equation, q* means the normal derivative of fundamental solution θ*. Using the diffusive fundamental solution, as presented earlier in equation (7), and the Dirac Delta function properties, it is possible to write:

$$\int_\Omega \Phi\theta\theta^*_{,ii} d\Omega = -\,c(\xi)\Phi(\xi)\theta(\xi) \tag{13}$$

In last equation, c(ξ) is the coefficient related to the position of field point ξ to be taken: internally, externally or then at the boundary of domain field X. The next step is the mathematical treatment of the third term in the right hand side of the integral equation (12) by the Quasi-Dual procedure. The purpose is to transform this domain integral into a boundary integral. The following approximation is required:

$$b_i = \Phi\,\theta_{,i} \approx \alpha^j_p \Psi^j_{p,i} = \alpha^j_p \eta^j_{pi} \tag{14}$$

Using the approximation given by equation (14) and integration by parts, it is results in:

$$\int_\Omega \Phi\,\theta_{,i}\theta^*_{,i} d\Omega = \alpha^j_p \int_\Omega \Psi^i_{p,i}\theta^*_{,i} d\Omega = \alpha^j_p \int_\Omega [\Psi^i_p \theta^*_{,i}]_{,i} d\Omega + \alpha^j_p \int_\Omega \Psi^i_p \theta^*_{,ii} d\Omega \tag{15}$$

Using divergence theorem and properties of the Dirac Delta function, it has:

$$\int_\Omega \Phi\,\theta_{,i}\theta^*_{,i} d\Omega = \alpha^j_p \int_\Gamma \Psi^i_p q^* d\Gamma - \alpha^j_p c(\xi)\psi(\xi) \tag{16}$$

The complete integral expression is then:

$$K\left\{c(\xi)\theta(\xi) + \int_\Gamma \left[\theta q^* - q\theta^*\right] d\Gamma\right\} = -\int_\Gamma v_i n_i \theta\theta^* d\Gamma + \int_\Gamma \Phi\,\theta q^* d\Gamma + c(\xi)\Phi(\xi)\theta(\xi) - \alpha^j_p\left[\int_\Gamma \Psi^i_p q^* d\Gamma + c(\xi)\Psi^j_p(\xi)\right] \tag{17}$$

In previous equation, q is the normal derivative of the temperature. Equation (17) can be discretized as usual by the collocation BEM formulation, the result of such procedure being a system of equations, which can be written in matrix form as:

$$\mathbf{H\theta - Gq + B\theta - H\Phi\theta = -\frac{1}{K} H\Psi\,\alpha} \tag{18}$$

Where next step is to eliminate the α vector in equation (18) using equation (19) below:

$$\boldsymbol{\alpha = \eta^{-1}[\Phi\theta_{,i}]} \tag{19}$$

The governing equation is a scalar one, but the source term b_i (equation (14)) taken separately is a vector. Therefore, it is necessary to put the interpolation function η in the dyadic form. The interpolation function can be chosen in a somewhat similar format as that used in the elastodynamic Dual Reciprocity procedure, such as Nardini and

Brebbia (1982). This dyadic type allows the inversion of matrix η, according to equation (19), as long as suitable functions are chosen, avoiding singularities. One such a class of functions is given by:

$$\eta^j_{pi} = 3RR_iR_p + R^3\delta_{ip} \tag{20}$$

Where $R=R(X_j;\ X)$ is the Euclidian distance between the interpolation point X_j and the field point X; δ_{ip} is the Kronecker delta operator, and:

$$R_p = [xp(X_j)-xp(X)] \tag{21}$$

Considering also the equation (20) end equation (14), it is easily demonstrated that:

$$\Psi^j_p = R^3R_p \tag{22}$$

Substituting the equation (19) in equation (18), it has:

$$\mathbf{H\theta - Gq + B\theta - H\Phi\theta = -\frac{1}{K}H\Psi\eta^{-1}[\Phi\theta,_i\,]} \tag{23}$$

There are still spatial derivative to be eliminated of the matrix system. The scheme adopted here is similar to that used by TDR. A new interpolation is done considering the following relation:

$$\boldsymbol{\theta} = \mathbf{F}\boldsymbol{\beta} \tag{24}$$

Thus, the spatial derivative of equation (24) results:

$$\boldsymbol{\theta},_i = \mathbf{F},_i\,\boldsymbol{\beta} \tag{25}$$

Considering:

$$\boldsymbol{\beta} = \mathbf{F}^{-1}\boldsymbol{\theta} \tag{26}$$

One has:

$$\boldsymbol{\theta},_i = \mathbf{F},_i\,\mathbf{F}^{-1}\boldsymbol{\theta} \tag{27}$$

The final matrix equation reads:

$$\mathbf{H\theta - Gq + B\theta - H\Phi\theta = -\left(H\Psi\ \eta^{-1}\varphi\theta,_i\,F,_i\,F^{-1}\right)\theta} \tag{28}$$

Taking the last equation in a more compact form:

$$\mathbf{H\theta - Gq + B\theta - S\theta = -M\theta} \tag{29}$$

In general, it is always possible to condense the former matrix system and to express it as:

$$\mathbf{[H + B - S - M]\theta = GQ} \tag{30}$$

From now on the classical procedures of BEM can be used without difficulties.

4. Examples

The examples discussed next show results obtained with the HQDR and the TDR formulations. For both approaches, the numerical results are compared with the analytical solution and against each other.

4.1 First Example: One-dimensional fluid flow

The first example consists of a one-dimensional fluid flow in a square control volume, where distinct temperatures are prescribed between vertical boundaries. The figure shows the physical features of the problem. The velocity field is constant along the horizontal direction. No fluxes and velocities exist in the vertical direction. The diffusion heat transfer processes itself in a counter current fluid flow. Regular meshes composed by 20,40,80,160 constant elements are employed in this simulation.

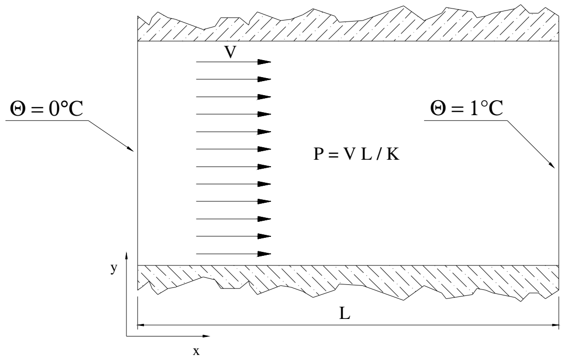


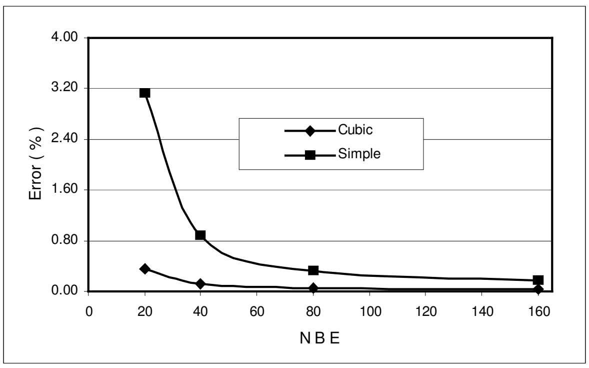
Figure 1. Physical and geometric features of first example.

The analytical solution for temperatures is given by equation (31), while the normal derivative in the horizontal direction is given by equation (32). In Both equations p is the quotient between the thermal conductivity and the uniaxial velocity.

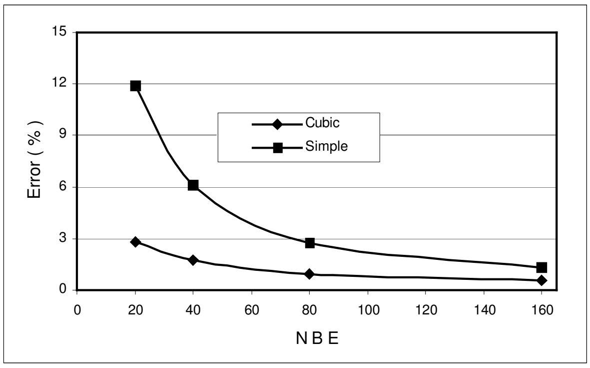
$$\theta = \frac{e^{px} - 1}{e^{pl} - 1} \tag{31}$$

$$q = \frac{p\, e^{px}}{e^{pl} - 1} \tag{32}$$

In the first test the performance of the radial function used to interpolate the spatial derivatives in F matrix is shown (equation (27)). It was detected that cubic radial functions are very superior than simple radial ones. The graphs in the figure (2) and (3) show this behavior for HQRD formulation. In these simulations the fluid flow has Peclet number 2. Figure (2) shows the percentage error curve of the temperatures along the nodal points of the horizontal face of the control volume considering different meshes. Figure (3) presents the same procedure analyzing the temperature derivative on the right vertical face. Both percentage error curves were obtained using analytical solution as reference.



Figures 2. Performance evaluation of the radial interpolation functions for the temperatures calculation



Figures 3. Performance evaluation of the radial interpolation functions for the temperature derivatives calculation

In the second simulation the performance of Hybrid Quasi-Dual Reciprocity and the traditional Dual Reciprocity are compared, in the same condition of the previous simulation: Peclet number is 2 and the influence of mesh refinement is evaluated. It is important to notice that the traditional Dual model employs 16 internal points (poles) to improve the results. No poles were requested in HQDR model. Figures (4) and (5) present the curves of percentage error in temperature on the horizontal faces and the normal derivative of the temperature on the right vertical face. Both graphs show that the performance of HQDR formulation is superior.

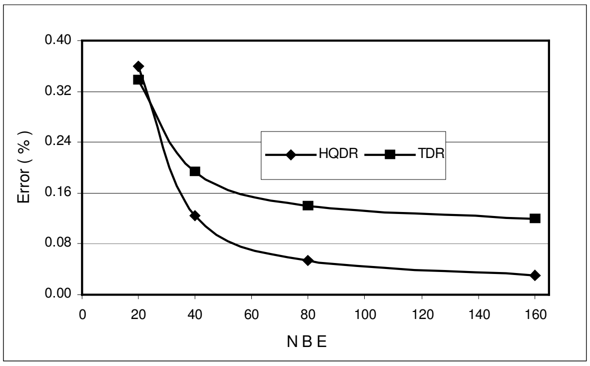


Figure 4. Influence of mesh refinement for the temperature

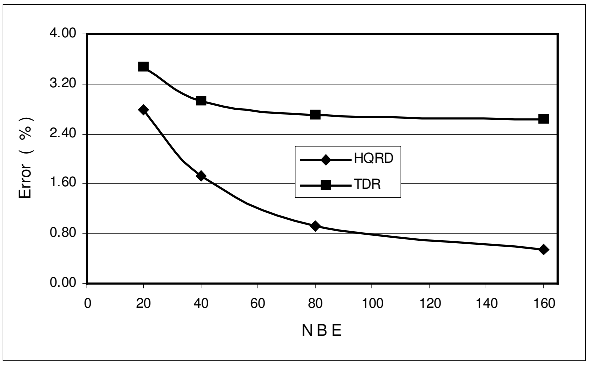
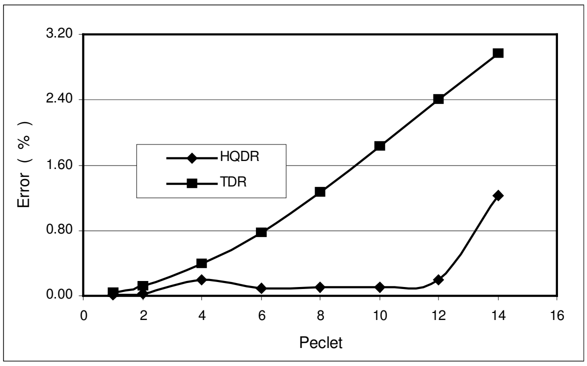
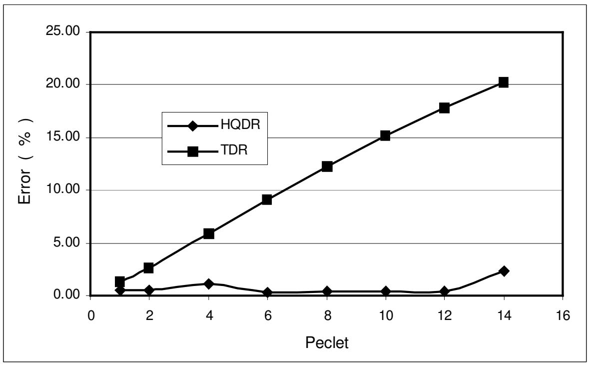


Figure 5. Influence of mesh refinement for the temperature derivative

The last test in this example analyses the performance of TDR and HQDR with the increase of Peclet number. One hundred sixty constant boundary elements were used to implement this experience. The percentage error curves for temperature and temperature derivative are presented in figures (6) and (7) respectively.



Figures (6). Influence of Peclet number for the temperature values



Figures (7). Influence of Peclet number for the temperature derivative values

4.2 Second Example: Two-dimensional fluid flow

The second example analyzed is a two dimensional problem, where the constant velocity field has components in x and y directions. Figure (8) presents the physical and geometrical features of this problem.

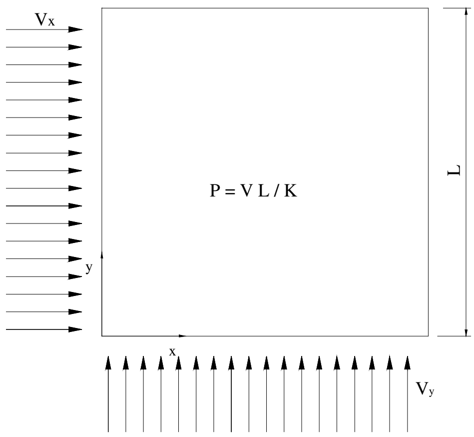


Figure (8). Features of the second example: square control volume with two dimensional velocity field

Only temperature is prescribed over the entire boundary according to the following expression:

$$\theta = e^{vx+wy} \tag{35}$$

For sake of simplicity, the velocity components v and w are assumed equal. Figures (9) and (10) present the curves of percentage error in numerical simulation for normal derivatives of temperature on the horizontal and vertical faces, respectively. In this test the Preclet number is 2 Average of percentage nodal errors were calculated for different meshes employing 20, 40, 80 and 160 boundary elements for both HQDR and TDR, but this last used 16 poles for improvement of the results.

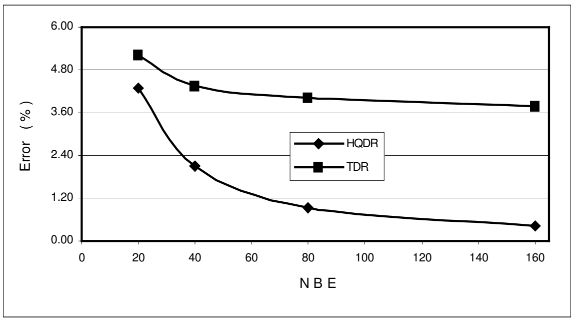


Figure (9). Percentage error in numerical simulation for normal derivatives of temperature on the horizontal face

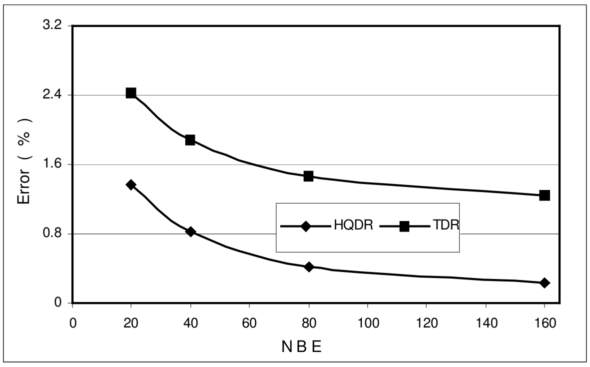


Figure (10). Percentage error in numerical simulation for normal derivatives of temperature on the vertical face

It can be realized the better performance of HQDR method. Although results in vertical faces are most important, the error in horizontal faces is larger. This can be explained due to the difficulty of Dual Reciprocity to represent values of different levels of meaning using the same global interpolation function. Fortunately, the highest values are better represented. For the next test the most refined mesh is used to evaluate the performance as a function of Peclet number. Figures (11) and (12) shown the results:

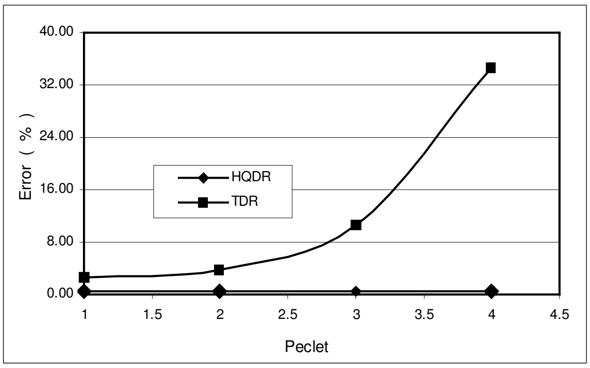


Figure (11). Numerical performance of both methods as a function of Peclet number. Horizontal face

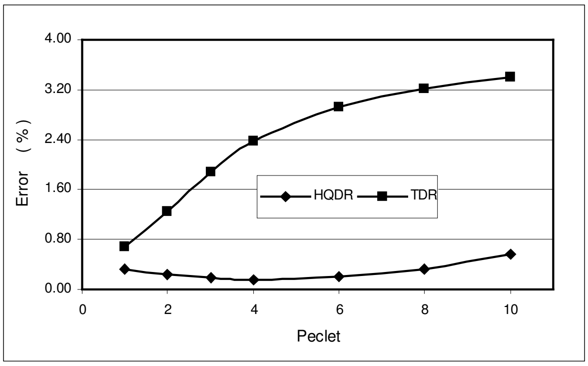


Figure (12). Numerical performance of both methods as a function of Peclet number. Vertical face

Accurate numerical solutions for two-dimensional problems are much harder to achieve that for one-dimensional ones. Especially for TDR, the results in the horizontal face were not very accurate because of the already mentioned difficulty to interpolate small values in global sense. However, the HQDR results are very superior on the vertical face too.

5. Conclusions

The recent Quasi-Dual Boundary Element formulation achieves very good results in convective heat transfer for potential fluid flow with median Peclet numbers, that is, when the convective phenomena is not absolute, independent of the velocity field being constant or variable. In spite of that the research of better boundary element formulations for this class of problems is still in process. For this reason Hybrid Quasi Dual Reciprocity Boundary Element formulation was proposed and new adaptations of Dual Reciprocity idea will be created in the future.

The rewriting of the integral term related to convective phenomena through four new terms was very successful, since good results were achieved in the several simulations implemented, two of them shown in this paper. The comparison with Traditional Dual Reciprocity is important, because both formulations use a specific interpolation to eliminate the spatial derivative. However, in HQDR this approximation occurs just in one of four integral terms, whereas in the TDR this interpolation interferes entirely in the convective integral term.

It is important to point out that poles were not required by the HQDR. This fact means, naturally, a reduction in the computational cost, but the most important implication is the absence of other parameter to control accuracy beyond the mesh refinement.

In synthesis, the HQDR formulation showed good accuracy and flexibility. This feature allows it to be used as an interesting option to solve advective problems beside other former boundary formulations, such as traditional Dual Reciprocity and the technique that uses diffusive-advective fundamental solution.

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