THERMIC TRANSIENTS IN AXISYMMETRIC BODIES MODELLING BY BOUNDARY ELEMENT METHOD

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Abstract. *This work presents a Boundary Element formulation solution to transient heat conduction in axisymmetric problems. In the methodology employed here the three-dimensional variables are transformed into two-dimensional ones through the Fourier Series procedure. An expansion in the angular spacial coordinate is made and the fundamental solution is integrated on the circumpherential direction resulting in a set of two-dimensional harmonic components, associated to each term of the Fourier Series. In the approach, the bodies can be submitted to general and arbitrary boundary conditions. The Dual Reciprocity Technique is used to transform the domain integral related to the transient effect into boundary integrals. Thus, the well-known features of the Boundary Element Method are sustained: the discretization is limited to the boundary of the revolution section and high accurate levels are obtained with relative few numbers of nodal points. The time marching process is implementd through the central Finite Difference scheme, where temperature and flux are calculated simultaneously in each time step of the transient response.*

Keywords: *Axisymmetric problems, Boundary Element Method, Transient heat conduction.*

1. Introduction

The application of the Boundary Element Method (BEM) to axisymmetric problems is very advantageous. Using axyssimetric fundamental solution for the integration of three-dimensional problems, BEM produces a very concise numerical model with reduced input data and low computational cost. Many researchers studied and proposed procedures to further improve the BEM application in this field, specially by producing very accurated numerical results. An elaborated review about this matter can be obtained in Kitbee (1995) and Brebbia et al (1984). Just to do mention a few of the most important authors, Rizzo and Shippy (1980), Cisternas, Telles and Mansur (1986) and Provatidis (1998) gave important contributions to stationary axisymmetric analysis. From them many ideas were used to elaborate this work.

It is very logical to take advantage of this features of the Boundary Element model to transient axisymmetric analysis. Wrobel at al (1986) was the first to approach this problem. His pioneer model, however, is specific to axisymmetric boundary conditions and few informations were given in his paper about the interesting numerical aspects that can arise on this subject.

In this work a most general formulation is presented, which allows to examinate axisymmetric bodies submited to boundary conditions without symmetry. Some of the several numerical features are exposed, mainly related to suitable time steps and poles effects. An important point, related to the behavior of solution in the initial period of transient response, is discussed. Two basic examples are solved. More general cases could not be presented by space limitations and absence of analytical solutions to evaluate the accuracy, which would request other numerical solutions for comparison, such as Finite Element solutions.

2. Mathematical Model

The physical problem related to the transient heat conduction in a homogeneous and isotropic media is governed by the following partial differential equation:

$$K\nabla^2 u = \dot{u} \tag{1}$$

In this last equation K is the thermal conductivity, u is the temperature, ∇^2 is the Laplacian operator and the dot means time derivative. Essential and natural boundary conditions can be prescribed on time respectively by Eq. (2) and Eq. (3), showed below:

$$u(X,t) = \bar{u}(t) \quad \text{on} \quad \Gamma_u \qquad t > 0 \tag{2}$$

$$q(X,t)=\frac{\partial u}{\partial n}=\bar{q}(t) \quad \text{on} \quad \Gamma_q \quad t>0 \tag{3}$$

In Eq. (2) Γ_u is the part of the boundary where essential conditions are prescribed. Complementary in Eq. (3) Γ_q represents the part of the boundary where natural conditions are known. X represents the three dimensional coordinates (x,y,z) and n is the external normal on the boundary. The time dependency of the transient problem requests initial conditions in whole physical domain $\Omega(X)$, that is:

$$u(X,t)=u(X,0)=u_0(X) \quad X\in\Omega \tag{4}$$

3. Boundary Integral Formulation

The starting point of the BEM formulation can be given by the stablishment of the integral sentence using the fundamental solution u* as an auxiliary function, that is:

$$K\int_\Omega \nabla^2 u u^* d\Omega = \int_\Omega \dot{u} u^* d\Omega \tag{5}$$

The Dual Reciprocity approach implies that the fundamental solution u* corresponding to the stationary diffusive problem, given by Poisson equation, must be used:

$$\nabla^2 u^* = -\Delta(\xi;X) \tag{6}$$

In Eq. (6) $\Delta(\xi;X)$ is the Dirac delta function, which represents a point source at $X=\xi$. In three dimensions, u* is:

$$u^*(\xi:X)=\frac{1}{4\pi r} \tag{7}$$

It is necessary to transform the Eq. (5) into an inverse integral form. Considering initially the left hand side of this equation, the mentioned purpose is easily achieved with the use of Green's theorems and integration by parts. The following integral equation results:

$$K[\int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma + \int_\Omega \nabla^2 u^* u d\Omega] = \int_\Omega \dot{u} u^* d\Omega \tag{8}$$

Where q* is given by:

$$q^* = \partial u^* / \partial n \tag{9}$$

Substituing Eq. (6) into Eq. (8) it has:

$$K[\int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma - c(\xi)u(\xi)] = \int_\Omega \dot{u} u^* d\Omega \tag{10}$$

The coeficient $c(\xi)$ is related to the position of source point ξ in the domain $\Omega(X)$. Brebbia et al (1984) offer more details about this subject.

4. Transformation to Axisymmetric Variables

It is important to use a new sistem of coordinates more suitable to describe the geometric shape of axisymmetric bodies (see Fig. 1). Cylindrical coordinates (R,θ,z) are the natural choice for this case. The radial distance $r(\xi;X)$ must be redefined in cylindrical coordinates, that is:

$$r(\xi;X)=[\varsigma_x^2+\varsigma_\xi^2-2\varsigma_x\varsigma_\xi\cos(\theta_x-\theta_\xi)+(z_x-z_\xi)]^{1/2} \tag{11}$$

For convenience, the previous expression can be expressed by following sentence:

$$r = R\sqrt{1 - M^2 \cos^2(\theta_\xi / 2)} \tag{12}$$

Where:

$$R = \sqrt{a + b} \tag{13}$$

$$a = \varsigma_\xi^2 + \varsigma_x^2 + (z_\xi - z_x)^2 \tag{14}$$

$$b = 2\varsigma_x \varsigma_\xi \tag{15}$$

$$M^2 = \frac{4\varsigma_x \varsigma_\xi}{R^2} \tag{16}$$

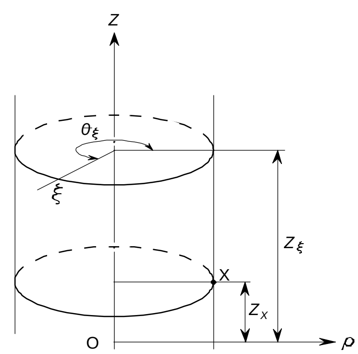

Figure 1. Cylindrical coordinates for an axisymmetric body.

The methodology employed in this paper uses the complex Fourier series. A simple axisymmetric analysis, in which the boundary conditions are also axisymmetric, did not request this tool. However, the procedure showed here is very general, allowing the approach of cases where the variable field is arbitrary. Therefore, the temperature and radial distance must be developed as a sequence of Fourier harmonics, that is:

$$u(\xi) = \sum_{n=0}^{+\infty} A_n(\rho_\xi, z_\xi) e^{in\theta_\xi} \tag{17}$$

$$u(x) = \sum_{n=0}^{+\infty} B_n(\rho_x, z_x) e^{in\theta_x} \tag{18}$$

$$\frac{1}{r(\xi, x)} = \sum_{n=0}^{+\infty} C_n(\rho_\xi, z_\xi; \rho_x, z_x, \theta_\xi) e^{in\theta_\xi} \tag{19}$$

In these last equations i means the complex unity. The coefficients A_n, B_n and C_n can be obtained by:

$$A_n(\rho_\xi, z_\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\xi) e^{-in\theta_\xi} d\theta_\xi \tag{20}$$

$$B_n(\rho_x, z_x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-in\theta_x} d\theta_x \tag{21}$$

$$C_n(\rho_\xi, z_\xi; \rho_x, z_x, \theta_x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{r(\xi, x)} e^{-in\theta_\xi} d\theta_\xi \tag{22}$$

Substituing the equations (20), (21) and (22) into Eq. (10) comes:

$$c(\xi)u(\xi) + \frac{1}{4\pi} \int_{\Gamma^*} B_n HL_n d\Gamma^* - \frac{1}{4\pi} \int_{\Gamma^*} H_n \frac{\partial B_n}{\partial n} d\Gamma^* = -\frac{1}{4\pi} \int_\Omega \dot{B}_n H_n d\Omega \tag{23}$$

Where:

$$d\Gamma^* = \rho_x d\Gamma \tag{24}$$

$$H_n(\rho_\xi, z_\xi; \rho_x, z_x) = \int_{-\pi}^{\pi} C_n(\rho_\xi, z_\xi; \rho_x, z_x, \theta_x) e^{in\theta_x} d\theta_x \tag{25}$$

$$HL_n(\rho_\xi, z_\xi; \rho_x, z_x) = \int_{-\pi}^{\pi} \frac{\partial}{\partial n} C_n(\rho_\xi, z_\xi; \rho_x, z_x, \theta_x) e^{in\theta_x} d\theta_x \tag{26}$$

At this point, the integral equation is a function of Fourier coefficients A_n ,B_n and C_n. The complete solution is achieved by superposition of all terms of the series. However, it is necessary to simplify the integral equation by elimination of common terms involving complex powers and to process the angular integration of axisymmetric fundamental solution and its normal derivative to each harmonic n. This very cumbersome task can be found in Manfrê et al (2002). Many procedures may be used to solve the resulting ellyptic integrals. In this paper Gauss quadrature is employed, taking advantage of its simplicity and good efficiency, according to Cisternas (1986).

5. Dual Reciprocity Procedure

In the right hand side of Eq. (23) exists a domain integral involving a time derivative of temperature. The use of the Dual Reciprocity approach allows to eliminate this domain integral by transforming it onto boundary integrals. For sake of simplicity, the three dimensional form of the domain integral, contrary to axisyimmetric form, must be considered. After the Dual Reciprocity procedures the axisymmetric integral equation can be taken again.

To implement this technique it is necessary to make initially the following aproximation in a temperature field:

$$\dot{u}(X,t) \approx \sum_{j=1}^{m} \dot{\alpha}^j(t) F^j(X) \tag{27}$$

This technique is similar to the separation of variables method; however, it uses a finite number of auxiliary functions α^j and F^j , defined in some points X^j, that can result in loss of accuracy. The main feature of Dual Reciprocity consists to find a primitive function F^j such as :

$$F^j(X^j; X) = \nabla^2 \Psi^j(X^j; X) \tag{28}$$

According to Wrobel (1986), an interesting arbitrary function is:

$$\Psi^j = r^3/12 \tag{29}$$

In the previous equation r is the distance between the interpolation points Xj and boundary points X. It is necessary to rewrite the Eq. (29) in cylindrical coordinates, that is:

$$\Psi^j = \frac{1}{12}[(R_j - R_k)^2 + (Z_j - Z_k)^2]^{3/2} \tag{30}$$

Determination of function F is done through the solution of the following differential equation:

$$F^j = \frac{\partial^2 \Psi^j}{\partial R^2} + \frac{1}{R}\frac{\partial \Psi^j}{\partial R} + \frac{\partial^2 \Psi^j}{\partial Z^2} \tag{31}$$

Solution of Eq. (31) results:

$$F^j = r\left(1 - \frac{R_j}{4R_k}\right) \tag{32}$$

The substitution of the harmonic function given by Eq. (29) into the transient integral term allows the aplication of the integration by parts technique and the divergence theorem, such as was done previously on the left hand side of Eq. (5). The resulting expression posseses only boundary integrals:

$$\int_{\Omega} \dot{u} u^* d\Omega \approx \dot{\alpha}^j [\int_{\Gamma} \frac{\partial \Psi^j}{\partial n} u^* d\Gamma - \int_{\Gamma} \Psi^j q^* d\Gamma - c(\xi)\Psi^j(\xi)] \tag{33}$$

It is interesting to define:

$$\eta^j = \frac{\partial \Psi^j}{\partial n} = \frac{r^2}{4}\frac{\partial r}{\partial n} \tag{34}$$

Adaptation on Eq. (33) considering Fourier series development results in:

$$c(\xi)u(\xi) + \frac{1}{4\pi}\int_{\Gamma^*} B_n H L_n d\Gamma^* - \frac{1}{4\pi}\int_{\Gamma^*} H_n \frac{\partial B_n}{\partial n} d\Gamma^* = -\frac{\alpha^j}{4\pi}[\int_{\Gamma} \dot{\eta}^j H_n d\Gamma - \int_{\Gamma} \dot{\Psi}^j H L_n d\Gamma - c(\xi)\Psi^j(\xi)] \tag{35}$$

Discretization of this last equation using traditional procedures of the Boundary Element Method, according Loeffler and Mansur (1986), produces axisymmetric H and G matrices that can be arranged in a following system:

$$GQ - HU = (G\eta - H\Psi)\dot{\alpha} \tag{36}$$

The next step consists to eliminate the α^j vector in Eq. (36), using the basic sentence of the Dual Reciprocity technique, exposed in Eq. (27). Then, it results in the following matrix equation:

$$(GQ - HU) = \frac{1}{K}(G\eta - H\psi)F^{-1}\dot{U} = C\dot{U} \tag{37}$$

The previous equation can be written simply by:

$$C\dot{U} + HU = GQ \tag{38}$$

6. Time Domain Discretization

The Eq. (38) is a time dependent matrix ordinary differential equation. The time discretization can be done easily using well known direct integration method, such as Finite Difference method, that is:

$$\dot{U} = \frac{U_n - U_{n-1}}{\Delta t} \tag{39}$$

This simple strategy has produced very good results in plane transient problems, which can be confirmed in many references, such as Loeffler and Mansur (1986) and Partridge et al (1992). For plane problems, the time step Δt usually obeys a special rule, given in many references such as Partridge et al (1992), but in axisymmetric problems the range of suitable values appears to be more sensitive. This will be demonstrated next.

The definitive matrix discretized form of Eq. (38) is given by:

$$(C + \Delta t H)U_n = CU_{n-1} + \Delta t G Q_n \tag{40}$$

In the previous equation the prescribed and unknown values of temperature and flux are interchanged by the original H and G matrices according to the traditional procedure of the Boundary Element Method to form an independent term of known values. Unknown values in future time steps are successively determined by the incremental scheme.

7. Examples

Two axisymmetric examples that have analytical solutions are solved in this paper. The boundary conditions prescribed are axisymmetric too in spite of the model capability to solve problems with general boundary conditions. The mesh employed in the numerical modelling uses constant elements. For sake of simplicity a special point in the domain is chosen for the analysis. The numerical evaluation of the temperature over time is ploted against the analytical response in some graphics where the effect of time step and level of mesh refinement can be examined.

7.1. First Example:Long Solid Cylinder

The physical and geometric features of this example are shown in the Fig. 2. The problem possesses also axial symmetry, because horizontal faces are isolated. That is, normal fluxes are prescribed and equal to zero. A sudden increase of the external value of temperature is imposed to the vertical face of the body, initially at zero degree. Then, all internal domain points undergo to a transient process over time until the body reaches the steady state equilibrium.

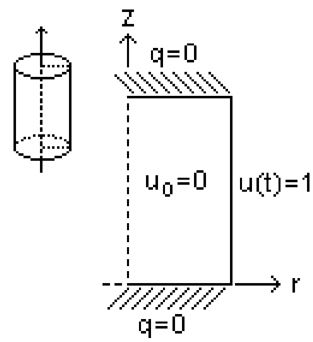

Figure 2. Long solid cylinder.

This problem does not present numerical difficulties to be solved. A simple mesh with 26 constant boundary elements produced good results. Considering this behavior, a first experience consists in the evaluation of the effect of poles in the numerical response. For this, the time behavior at the central point of the cylinder was analised using two meshes: the first has no poles and the second has 16 poles, both meshes possessing 26 constant boundary elements.

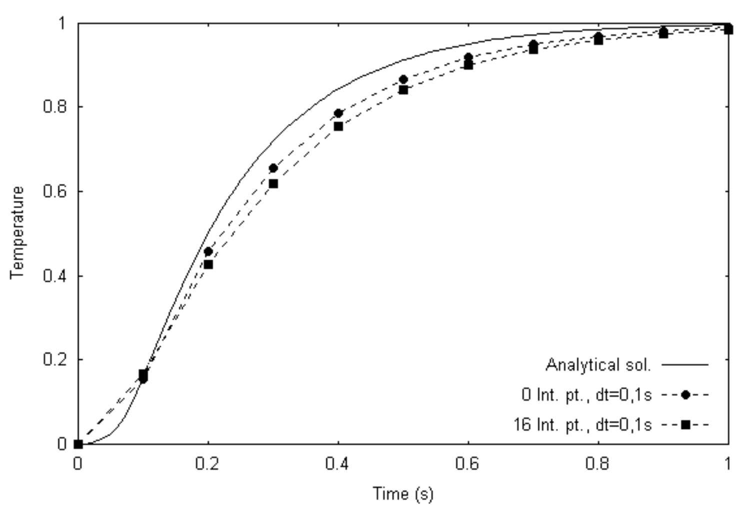

Figure 3. The poles effect in a mesh with 26 nodes when the time step is equal 0,1s.

The results of this test were very good, even if relatively few boundary nodes are employed, as in this simulation. The reason for this must be related to axial symmetry. The absence of poles did not affect this solution to a high degree, but their introduction marginally improved the quality of results. But the use of poor meshes is limited by the range of suitable time steps. Smaller time steps are requested for poor meshes.Unfortunatelly, small values of time step strongly excited the high modes of transient response. High modes representation usually is not good in the discrete analysis. Then, using a few boundary elements, the choice of time step is more important. The next figure demonstrated the effect of the smaller value of time step in the transient response:

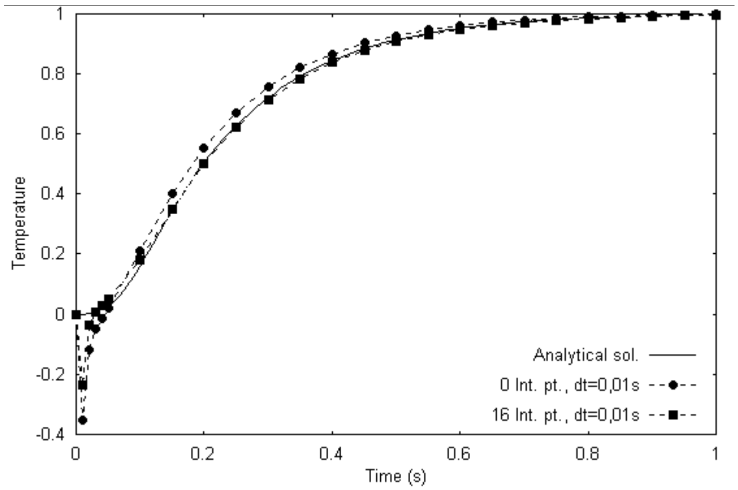

Figure 4. The poles effect in a mesh with 26 nodes when the time step is equal 0,01s.

The initial range of response is perturbed by high modes effect, as mentioned before. After this lapse of time, the agreement between analytical and numerical results are very good. The sole introduction of poles is not efficient to correct this problem; only the use of richer meshes and a suitable time step allow a correct representation of the initial period of transient process.

7.2. Second Example: Prolate Spheroidal Solid

The geometric characteristics of this example is depicted in the Fig. 5. Once more, a unit thermal shock is imposed to the external face of the solid at initial time. The internal domain of spheroid is at zero degree. The symmetry of spheroid to r axis has been taken into consideration.

Differently from the previous problem, this example presented higher numerical difficulties. To achieve good results in the initial time interval using a small time step, it was necessary to use 64 boundary elements with 9 poles. In this case, an oscilatory behavior can be noticed when the time step is not suitable. Figures 6 and 7 show the response of temperature at the center of the spheroid.

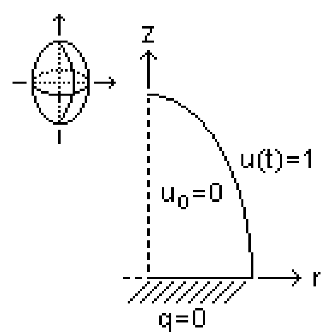

Figure 5. Geometric model for prolate spheroidal solid.

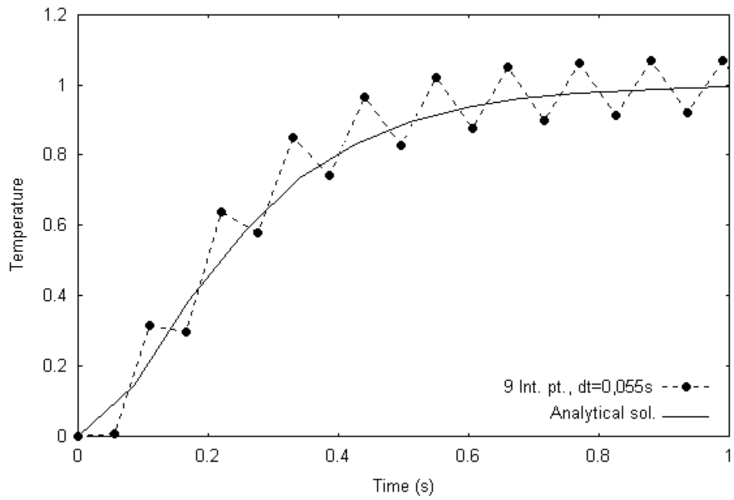

Figure 6. Mesh with 64 nodes, 9 poles and time step equal 0,055s.

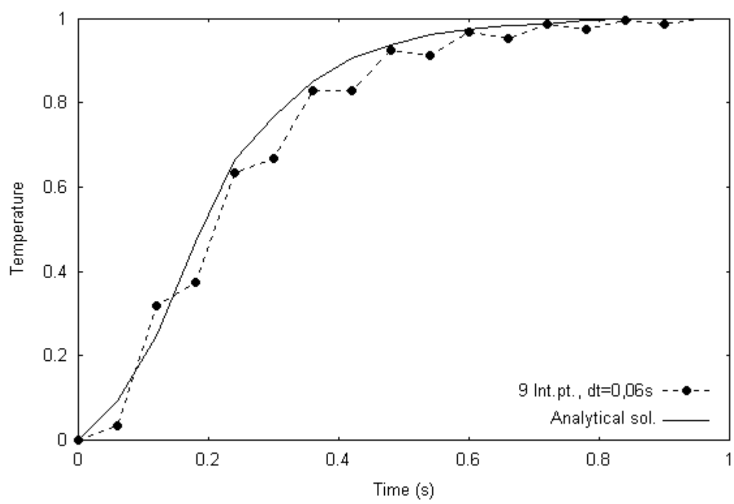

Figure 7. Mesh with 64 nodes, 9 poles and time step equal 0,060s.

An increase of the number of poles produces sensible improvement of the numerical results, such as can be noticed in the plott in Fig. 8, where can be seen a very good agreement between numerical and analytical solutions. But there is a limit to the poles power. Considering the degree of refinement of the boundary element mesh, a sole increase in the quantity of poles (higher than 15) does not produce better results.

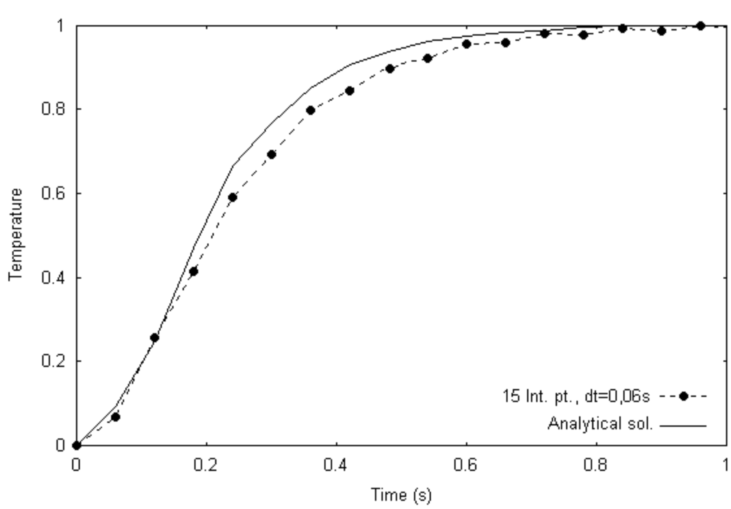

Figure 8. Mesh with 64 nodes, 15 poles and time step equal 0,060s.

8. Conclusions

The Boundary Element modeling to axisymmetric problems is very advantageous to engineering applications, because both the quantity of input data and the computational cost of processing are so very reduced. Stationary analysis done by many researchers showed excelent level of accuracy. Therefore, the disposament of this advantageous features to transient models would be very interesting. Unfortunally, the classical formulation, which uses the fundamental transient solution to axisymmetric problems, is very difficult to implement. This fact makes this strategy not attractive to professionals dealing with numerical modeling. The Dual Reciprocity formulation appears thus, as the most interesting way to apply the Boundary Element methodology in this class of problems.

The results obtained with the simulations accomplished were encouraging. The computational implementation was simple and the cost was very low. The accuracy can be considered very reasonable, since constant boundary elements were employed. However, an important point is the necessity to avoid very reduced time steps. Usually a discrete model requests very reduced time steps. In opposite, for axisymmetric applications the Dual Reciprocity seems to request an intermediary value of the time step situated between a minimum value to avoid the influence of high transient modes and a maximum to value to allow the accuracy of the numerical solution.

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