PERTURBATION OF A STATIONARY SOLUTION OF A NONLINEAR CONSERVATIVE SYSTEM UNDER RESONANCE CONDITIONS

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Abstract. *This work develops a method of perturbation of a stationary solution of a nonlinear conservative system. In the reference solution the frequencies depend on the momenta, that is, the fundamental frequencies of the reference solution are amplitude dependent and for some values of the momenta are linearly dependent. The basic assumption is that the Hamiltonian of the system can be expanded in terms of an n-dimensional convergent Fourier Series with the usual property that the magnitude of the coefficients decreases quadratically with the increase in frequency of the angular variables associated. The Hamiltonian is reduced to a simple form by means of a number of canonical transformations. The method is applied to a nonlinear oscillator and to the motion of a geostationary artificial satellite.*

Keywords*. Perturbation. Stationary Solution. Nonlinear System.*

1. Introduction

Behavior of Dynamical Systems where internal or external resonance is present is in general difficult to predict since it includes situations of isle formations in phase space or even chaotic patterns. This has been known for many years (e. g. Danby, 1970; 1971; Thompson and Stewart, 1989). A special Symposium on Periodic Orbits, Stability and Resonances was held at the University of São Paulo in 1969 (Giacaglia, 1970a) where several discussions arose during presentations of papers. Not mentioning typical mathematical studies or well known phenomena in electrical systems, a new interest on problems involving resonance was brought about with the development of space activities, when cientists, looking back at classical works by Poincaré (1957), Birkhoff (1927), von Zeipel (1916-17) and many others, discovered new ways of approaching resonance present in Satellite Theory, as is the case of critical inclination (Hori, 1960) or geosynchronous orbits (Kaula, 1966). Most of these works were devoted to systems where a small divisor appeared in methods of solution based on the averaging over fast variables of the system. In 1968, at a meeting of the Instituto Nazionale di Alta Matemática, this author presented a method of dealing with a more complex situation (Giacaglia, 1970b) where a two degrees of freedom system, presenting two independent resonance conditions, was reduced to a system with a single degree of freedom. Allan (1970) at the same meeting, studied an equally complex problem involving orbital resonance among natural satellites. A general theory based on averaging method was developed by this author (Giacaglia, 1970c) showing how multiple resonance situations could be represented by asymptotic series, in the sense of Poincaré (op. cit., Vol. I). Several aspects of resonance problems were discussed by this author (Giacaglia,1972) and received substantial addition in a later translation (Giacaglia and Markeeva, 1979). At a meeting at The University of Texas, this author (Giacaglia, 1979) developed a novel method of asymptotic series development in the presence of two small divisors in satellite motion. Later works introduced no novel approaches to the general problem of resonance, except for the established agreement that in many instances problems involving resonance could lead to a chaotic behavior, as shown by numerical integration of systems even with a small number of degrees of freedom. On the other hand, asymptotic series representing resonance situations in the motion of both natural and artificial celestial bodies have shown an excellent agreement with observations, for very long periods of time. Of course one has to consider, in this respect, some positive results by Kolmogorov (1952), Arnol´d (1963) and Moser (1962), dealing with the conservation of integral manifolds under the presence of perturbations. The role of dominant terms in the Fourier Series representing a properly reduced dynamical system was first shown by Garfinkle (1970). More recent works compare theoretical results with numerical evaluations, showing the existence of chaotic behavior of certain dynamical systems (Nesvorný and Morbidelli, 1998; Gozdziewski and Maciejewski, 1998). Analytical methods of perturbations of integrable canonical systems have been applied to problems of Celestial Mechanics, where numerical computations showed good agreement, have been applied by Grau and Gonzales (1999), Grau and Noguera (1999) and Gomes (1998) among several others. Nan and Luo (1998) applied analytical methods of perturbation to a special time dependent Duffing Oscillator exibiting resonance and chaos. Worth to be commented is a work by Butcher and Sinha (1998) where they applied canonical transformation theory dating back to Von Zeipel (op. cit) to a Mathieu-Hill equation, redescovering and renaming well established expansion techniques in Celestial Mechanics. As a colleague of mine well remembered, after Poincaré very few original theories have been developed. Amazingly enough, works by Brouwer (op. cit.), Hori (op. cit.) and many other scientists who worked in Satellites Theories are being forgotten at a very fast pace. In a recent work, this author (Giacaglia, 1999) has shown how to construct periodic orbits in the vicinity of a stationary point of a conservative system by applying a convergent method of successive approximations. It is shown in this work that stationary solutions and resonance situations are equivalent problems in well defined phase-spaces.

2. Definition of the Problem

The Hamiltonian Function H (q, p) is considered to be real analytic in an open set of $\mathbf{R}_{2n}$ and 2π-periodic in all angular variables q. Limiting the norm $|\nu|$ of each set of integers ν by an upper bound, the Fourier series

$$H(p,q) = A_0(p) + H_1(p,q) = A_0(p) + \sum_{|\nu|\neq 0} [A_1^{\nu}(p)\cos(\nu^T q) + B_1^{\nu}(p)\sin(\nu^T q)] \tag{1}$$

will be composed of a finite number of terms of the form $A_1^{\nu}(p)\cos(\nu^T q) + B_1^{\nu}(p)\sin(\nu^T q)$ for $|\nu| < K$, $0 < K$, finite.

The reference frequencies corresponding to $A_0(p)$ are defined by the $\omega_k(p) = A_{0p_k}(p)$. Resonance here is defined when there exists a point P_0 such that the reference frequencies are linearly dependent, that is, there is one set of n integers j_k such that $j^T\omega(p_0) = 0$. Assuming all momenta p to be present in A_0 a new set of canonical variables is introduced by the following transformation:

$$\begin{aligned} &y_i = q_i \quad (i = 1,2,\ldots,n-1),\ \ y_n = j_1 q_1 + j_2 q_2 + \cdots + j_n q_n \\ &p_i = x_i + j_i x_n \quad (i = 1,2,\ldots,n-1),\ \ p_n = j_n x_n \quad (j_n \neq 0),\ \ j^T\omega_0 = 0 \end{aligned} \tag{2}$$

The new reference frequencies are given by $\Omega(x) = A_{0\mathbf{x}}$ and one easily verifies that

$$\Omega_k(x) = \omega_k(p) \qquad (k = 1,2,\ldots,n-1),\ \ \Omega_n(x) = j^T\omega(p) \tag{3}$$

It follows that at $p = p_0$, all new frequencies Ω_k are different from zero and linearly independent, except for Ω_n which is zero. The problem is therefore equivalent to one where $A_{0x_{0n}} = 0$ for some value of $x = x_0$ and $A_{0x_{0i}} \neq 0$ for I = 1, 2, 3, …, n –1. In order to identify the real meaning of this, consider a trigonometric argument $\theta = \nu_1 q_1 + \nu_2 q_2 + \cdots + \nu_n q_n$. By changing to the new variables, one finds that

$$\theta = (\nu_1 - j_1\nu_n / j_n)y_1 + \cdots + (\nu_{n-1} - j_{n-1}\nu_n / j_n)y_{n\ 1} + (\nu_n / j_n)y_n \tag{4}$$

Suppose now that the argument θ is such that $j \,//\, \nu$, where $j^T\omega_0 = 0$. This is called a critical argument. It follows that this argument is reduced to the simple form $\theta = (\nu_n / j_n)\, y_n$. The critical argument at P_0 is reduced to a rational multiple of y_n alone. Since for $x = x_0$, $\Omega_0(x_0) = 0$, it follows that the angle y_n is stationary. One can state that a single resonance among the reference frequencies of a system leads to a stationary motion in a conveniently defined space of new variables. Vice-versa, if for some value of $x = x_0$, one of the reference frequencies of the system is zero, this corresponds to resonance (linear dependence) among the frequencies of the system in a conveniently defined space of new variables.

2.1. Development of the Solution

In order to study the motion of the system when p is close to p_0, that is when x is close to x_0, consider the transformed Hamiltonian

I) $H = A_0(x) + H_1(x, y)$

II) $H_1 = \sum_{\nu}[A_1^{\nu}(x)\cos(\nu^T y) + B_1^{\nu}(x)\sin(\nu^T y)]$, and for $|\nu| < K_\nu$ finite, $|A_1^{\nu}(x)|^2 + |B_1^{\nu}(x)|^2 < CK_\nu^{-4} = O(\varepsilon^{2k})$, $C > 0$, $0 < |x| < X$, $K_\nu = O(\varepsilon^{-k/2})$

III) For $x = x_0$, $A_{0x_i}(x_0) = \omega_{0i} \neq 0$, $i = 1, 2, \ldots, n-1$ and $A_{0x_n}(x_0) = \omega_{0n} = 0$

IV) The motion is to be determined for values of $x \approx x_0$, such that

$$A_{0x_n}(x) = \omega_n(x), \quad 0 < |\omega_n(x)| < \delta\varepsilon^s, \ \ 0 < \delta\text{finite}, 0 < \varepsilon < 1, s > 0.$$

The problem is to eliminate as many angular variables as possible from the Hamiltonian by a suitable canonical transformation to new variables (ξ, η) generated by the Jacobi function

$$S(\xi, y) = \xi^T y + W(\xi, y) \tag{5}$$

such that $x = \xi + W_y$, and $\eta = x + W_\xi$ where $W(\xi, y)$ is a Fourier series similar to $H(p,q)$ and the maximum value of $|W|$ is of some order ε^r $(r>0)$. It will be shown that the order of magnitude r depends on s and on the order of magnitude of H_1 containing isolately the critical argument y_n.

The new Hamiltonian $K(\xi, \eta)$ in the new variables (ξ, η), takes the usual (Brouwer and Clemence, 1961) expanded form $K(\xi, \eta) = K_0(\xi) + F(\xi, \eta)$, a Fourier series similar to $H(p, q)$. Expanding in Taylor series both sides Eq. 6 below

$$H(y, \xi + W_y) = K(y + W_\xi, \xi) \tag{6}$$

it is found that

$$\begin{aligned} &A_0(\xi) + A_{0\xi} W_y + \tfrac{1}{2!} A_{0\xi\xi} W_y^2 + \cdots + H_1(y, \xi) + H_{1\xi} W_y + \tfrac{1}{2} H_{1\xi\xi} W_y^2 + \cdots \\ &= K_0(\xi) + F(y, \xi) + F_y W_\xi + \tfrac{1}{2!} F_{yy} W_\xi^2 + \cdots \end{aligned} \tag{7}$$

Matching terms of same order of magnitude in ε one finds $K_0(\xi) = A_0(\xi)$. Next consider the following cases

I) $H_1(y, \xi)$ does not contain the critical argument y_n isolately. In this case, the equation

$$\sum A_{0\xi_i} w_{y_i} + H_1(y, \xi) = F(y, \xi) \tag{8}$$

is of first order if W is of first order and so is also F, that is, $W = S_1 + S_2 + \cdots$ and $F = K_1 + K_2 + \cdots$. The term $A_{0\xi_n} W_{y_n}$ is order r+s, that is, order 1+s in ε and is included since it does not introduce small divisors. In fact, defining

$$K_1(\xi) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} H_1(y, \xi) dy_1 \cdots dy_{n-1} \tag{9}$$

it follows that

$$\sum_{1=1}^{n} A_{0\xi_i} S_{1y_i} + \hat{H}_1(y, \xi) = 0 \tag{10}$$

where

$$\hat{H}_1 = \sum_\nu [A_1^\nu(\xi) \cos(\nu_1 y_1 + \cdots \nu_n y_n) + B_1^\nu(\xi) \sin(\nu_1 y_1 + \cdots \nu_n y_n)] \tag{11}$$

and $\nu_1, \nu_2, \ldots, \nu_{n-1}$ are not all zero. Therefore, it is found that

$$S_1 = \sum_\nu -[A_1^\nu(\xi)/A_{0\xi}\nu] \sin(\nu_1 y_1 + \cdots \nu_n y_n) + \sum_\nu [B_1^\nu(\xi)/A_{0\xi}\nu] \cos(\nu_1 y_1 + \cdots \nu_n y_n) \tag{12}$$

The process can be repeated up to any order of approximation. If there is a term containing y_n alone, another process must be used for in this case the denominators $(A_{0\xi}\nu)$ will be of order ε^s when $\nu_1 = \nu_2 = \cdots = \nu_{n-1} = 0$.

II) $H_1(y, \xi)$ does contain terms where y_n appears isolately. In this case one may write this term as

$$H_1(y, \xi) = \sum_\nu [A_1^\nu(\xi) \cos(\nu^T y) + B_1^\nu(\xi) \sin(\nu^T y)] + \sum_\alpha [A_1^\alpha(\xi) \cos \alpha y_n + B_1^\alpha(\xi) \sin \alpha y_n], \ \alpha \geq 1 \tag{13}$$

and $\nu_1, \nu_2, \ldots, \nu_{n-1}$ are not all zero in any term. Initially, all angular variables, except y_n, are eliminated by defining

$$K_1(\xi, y_n) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} H_1(y, \xi) dy_1 \cdots dy_{n-1} \tag{14}$$

so that the resulting Hamiltonian becomes

$$K(\xi, \eta_n) = K_0(\xi) + K_1(\xi, \eta_n) + K_2(\xi, \eta_n) + \cdots \tag{15}$$

The resulting system has a single degree of freedom and can be reduced to a quadrature. This integration can generrally be performed by an asymptotic series as follows.

Consider the Hamiltonian $K(x, y) = K_0(x) + K_1(x, y) + K_2(x, y) + \cdots$ where x and y are scalar variables of a system with a single degree of freedom, $x = \xi_n$, $y = \eta_n$, all other momenta ξ_k , $k = 1, 2, ..., n-1$, being constant parameters corresponding to the ignorable angular variables η_k , $k = 1, 2, ..., n-1$. It has been observed that $K_{0x} = 0(\delta\varepsilon^s)$. One looks for a canonical transformation to new variables (x', y') generated by the function

$$S(x', y) = x'y + W(x', y) \tag{16}$$

such that the transformed Hamiltonian is a function only of the new momentum x´.

Expanding the energy equation $K[x(x', y'), y(x', y')] = K'(x')$ and assuming, as usual, that $K'_0(x') = K_0(x')$, by writing $K' = K'_0 + F'(x')$, it is found that

$$K_{0x'}W_y + \tfrac{1}{2!}K_{0x'x'}W_y^2 + \cdots + K_1(x', y) + K_{1x'}W_y + \cdots + K_2(x', y) + \cdots = F'(x') \tag{17}$$

If W is O(ε^r), the following equations hold true

$$\begin{aligned} &K_{0x'}W_y = O(\varepsilon^{r+s}), \quad K_{0x'x'}W_y^2 = O(\varepsilon^{2r}), \quad K_1 = O(\varepsilon^1), \\ &K_{1x'}W_y = O(\varepsilon^{r+1}), \quad K_2 = O(\varepsilon^2), \quad F'(x') = \text{order to be specified} \end{aligned} \tag{18}$$

If W_y has to be not identically zero and furthermore always determined even if $K_{0x'} = 0$, then $r + s \geq 1,\ 2r = 1,\ F'(x') = O(\varepsilon)$, so that $r = 1/2, s \geq 1 - r$. If s < 1/2 the situation is not critical and for the values of x´ giving $K_{0x'} = O(\varepsilon)$ the problem may be treated as of non resonance. It should be noted that it has been assumed $K_{0x'x'}$ not to be small, that is, it should be O(ε^0). On the other hand, if $K_p(x, y)$ is the lowest part of K which contains y, that is,

$$K(x, y) = K_0(x) + K_1(x) + \cdots + K_{p-1}(x) + K_p(x, y) + \cdots \tag{19}$$

then the relations above become $r + s \geq p,\ 2r = p,\ F'(x') = O(\varepsilon^p)$, so that $r = p/2,\ s \geq p/2$ and one has a narrower region of resonance since for p > 1 / 2, it follows that $\varepsilon^{p/2} < \varepsilon^{1/2}$. For any order of approximation the recurrence equation assumes the form

$$AW_{ky}^2 + BW_{ky} + F_k(x', y) = K'_k(x') \tag{20}$$

where $W = W_r + W_{r+s/2} + W_{r+s} + \cdots$

The usual (Hori, 1960) approach to the solution of this equation is to define

$$K'_k(x') = \min_y F_k(x', y) = F_k[x', y_0(x')] \tag{21}$$

where it should be mentioned that F_k is made up of $K_k(x',y)$ plus terms $O(\varepsilon^k)$ arising from lower order terms and therefore depends on the definition of K'_p and W_p for p=s, s+s/2, 2s, …, k-s/2. Therefore, the function $\varphi_k(x',y) = F_k(x',y) - K'_k(x')$ is positive everywhere but at y = y_0 where it is zero. All of the above assumes that $F_k(x',y)$ has a minimum at y = y_0 . The stationary solution in the (x,y) plane is (x_0,y_0) where $K_{0x}(x_0) = \Omega(x_0) = 0$.

2.2. Reduction to the Ideal Resonance Problem.

In the previous sections it was shown how to reduce the Hamiltonian of the system to a form that may be written as

$$K(\xi_1,\xi_2,...,\xi_n,\eta_n) = A_0(\xi_1,\xi_2,...,\xi_n) + \sum_{\nu\neq 0} A_\nu(\xi_1,\xi_2,...,\xi_n)\cos\nu\eta_n + B_\nu(\xi_1,\xi_2,...,\xi_n)\sin\nu\eta_n \quad (22)$$

and the momenta $\xi_1,\xi_2,...,\xi_{n-1}$ are certainly constants because the corresponding angular variables are ignorable.

Therefore, hiding all constant parameters the reduced Hamiltonian is written as

$$K(\xi,\eta) = A_0(\xi) + \sum_{k\neq 0} A_k(\xi)\cos k\eta + B_k(\xi)\sin k\eta \quad (23)$$

where the transformed Fourier series has the same properties of the original one, that is, the coefficients $A_k(\xi)$ and $B_k(\xi)$ are at most $O(\varepsilon)$.

It can be shown (Giacaglia, 1970d) that the Hamiltonian can be reduced to the ideal resonance problem by means of a canonical transformation generated by a function $S(\xi, y) = \xi y + S_{1/2} + S_1 + S_{3/2} + \cdots$. The Hamiltonian assumes the new simple form

$$H(x,y) = K[\xi(x,y),\eta(x,y)] = P(x) + Q(x)\cos y + R(x)\sin y \quad (24)$$

This has been defined as the Ideal Resonance Problem by Garfinkel (1970) and has well known properties.

The consideration of special conditions define whether one has libration or circulation around a center in the phase plane (ξ, η) or an asymptotic trajectory toward a saddle point (Giacaglia, 1970d).

2.3. Application to a nonlinear oscillator

Let us consider the problem defined by the Hamiltonian

$$\begin{aligned} H(p,q) = \tfrac{1}{2}(p_1^2 + p_2^2) + A_1^{1,1}(p)\cos(q_1+q_2) + A_1^{2,-1}(p)\cos(2q_1-q_2) + \\ + A_2^{2,2}(p)\cos(2q_1+2q_2) + A_2^{4,-2}(p)\cos(4q_1-2q_2) \end{aligned} \quad (25)$$

Suppose the following values for the reference orbit defined by the very first term of H:

$$p_{10} = 1 = \omega_{10},\; p_{20} = 2 = \omega_{20} \quad (26)$$

Corresponding to these values, resonance occurs corresponding to the critical argument $2q_1$-q_2 and all of its multiples. According to Eqs. (2), (3) and (4) the Hamiltonian takes the form

$$\begin{aligned} H(x,y) = \tfrac{1}{2}(x_1+2x_2)^2 + \tfrac{1}{2}x_2^2 + A_1^{1,1}(x)\cos(3y_1-y_2) + A_1^{2,-1}(x)\cos y_2 + \\ + A_2^{2,2}(x)\cos(6y_1-2y_2) + A_2^{4,-2}(x)\cos 2y_2 \end{aligned} \quad (27)$$

It is seen that the critical argument has been reduced to a single variable y_2. According to Section 2.1, assuming the definitions

$$A_0 = \tfrac{1}{2}(x_1 + 2x_2)^2 + \tfrac{1}{2}x_2^2,\ \ H_1 = A_1^{1,1}(x)\cos(3y_1 - y_2) + A_1^{2,-1}(x)\cos y_2$$
$$H_2 = A_2^{2,2}(x)\cos(6y_1 - 2y_2) + A_2^{4,-2}(x)\cos 2y_2 \tag{28}$$

Straightforward developments lead to the results

$$K_0(\xi_1,\xi_2,\eta_2) = \tfrac{1}{2}(\xi_1 + 2\xi_2)^2 + \tfrac{1}{2}\xi_2^2 + \tfrac{1}{40}\xi_2(\xi_2 - \xi_1)$$
$$K_1(\xi_1,\xi_2,\eta_2) = \tfrac{3}{16}\xi_1\xi_2\cos\eta_2, \quad K_2(\xi_1,\xi_2,\eta_2) = \tfrac{1}{100}\xi_1^2\cos 2\eta_2 \tag{29}$$

In all previous equations it has been assumed that

$$A_1^{1,1} = -\tfrac{1}{2}p_2(p_1 + p_2),\ A_1^{2,-1} = -\tfrac{1}{5}p_2(p_1 + 2p_2),\ A_2^{2,2} = \tfrac{1}{40}p_2^2,\ A_2^{4,-2} = \tfrac{1}{100}(p_1 + 2p_2)^2 \tag{30}$$

According to the procedure pointed out in Section 2.2, reduction to the ideal resonance problem via a Jacobi generating function is possible, together with the definition of the singular points in the transformed phase plane (x,y) of (ξ_2, η_2).

If one defines the reduction as

$$K(\xi,\eta) = \sum_{j\ge 0} A_j(\xi_1,\xi_2)\cos j\eta_2 = K(\xi(x,y),\eta(x,y)) = Z(x,y) = P(x) + Q(x)\cos y \tag{31}$$

the first approximation to the generating function $S(\xi, y) = \xi y + S_{1/2} + \cdots$ of the necessary transformation is given by Eq.32

$$A_{0\xi}S_{1/2,y} + \tfrac{1}{2}A_{0\xi\xi}S_{1/2,y}^2 + \sum_{j\ge 0} A_j(\xi)\cos jy = P_1(\xi) + Q_1(\xi)\cos y \tag{32}$$

where ξ stands for ξ_2 and y is the new angular variable.

Introducing the values for A_0 it is found that

$$S_{1/2,y} = -(\tfrac{23}{34}k + \xi) \pm \{(\tfrac{23}{34}k + \xi)^2 + \tfrac{8}{17}[2A_1\cos^2(y/2) - \sum_{j\ge 0} A_j(\xi)\cos jy]\}^{1/2} \tag{33}$$

At y = 0 the right hand side of Eq. 33 takes the value

$$S_{1/2,y}(y=0) = -(\tfrac{23}{34}k + \xi) \pm \{\Delta\}^{1/2} \tag{34}$$

$$\Delta = (\tfrac{23}{34}k + \xi)^2 + \tfrac{8}{17}[2A_1 - M(\xi)], \quad M(\xi) = \sum_{j\ge 1} A_j(\xi) \tag{35}$$

If $\Delta > 0$, $S_{1/2,y}$ is always real and therefore y undergoes circulation. If $\Delta < 0$, $S_{1/2,y}$ becomes complex at some value away from y = 0 and this value will never be reached. In this case one has libration around the point of minimum $y = \pi$.

At $y = \pi$ the right hand side of Eq. 33 takes the value

$$S_{1/2,y}(y=0) = -(\tfrac{23}{34}k + \xi) \pm \{\Delta\}^{1/2} \tag{36}$$

$$\Delta = (\tfrac{23}{34}k + \xi)^2 - \tfrac{8}{17}\sum_{j\ge 1}(-1)^j A_j(\xi) \tag{37}$$

Suppose now that the additional condition expressed by Eq. 38 is satisfied

$$\sum_{j\ge 1}(-1)^j A_j(\xi) = m(\xi) = 0 \tag{38}$$

the result at $y = \pi$ would be

$$S_{1/2,y}(y=\pi)=0 \tag{39}$$

A possible situation would be for $S_{1/2}$ to be a sine series in y, but this might not be the case. In any event, at y = π, $S_{1/2}$ is an arbitrary function of ξ. On the other hand, the differential equation for y is given by

$$\dot{y}=\partial Z(x,y)/\partial x = A_{0x}+\sum_{j\geq 1} A_{jx}\cos jy \tag{40}$$

and at y = π the result is that $\dot{y}(y=\pi)=A_{0x}+\sum_{j\geq 1}(-1)^j A_{jx}$. Under the assumption that A_{jx} are order of magnitude less than A_{0x}, one concludes that a good approximation to the libration point is given by $A_{0x}=0$ and y = π.

2.4. Geostationary Artificial Satellite

A classical problem of resonance corresponds to the equations of motion of an artificial satellite at an altitude of approximately seven earth radii leading to an orbital period of 24 hours. Any axial asymmetry of the earth potential field will generate a situation of resonance between these anomalies and the motion in longitude of the satellite. In this case, the relevant part of the Hamiltonian of the system, using appropriate variables, may be written as

$$H(p,q)=H_0(p_1,p_2)+H_1(p_1,q_1+q_2) \tag{41}$$

The relative orders of magnitude of these two terms, computed by the values of the physical constants representing the earth potential field are $|H_0(p)|=O(1), |H_1(p,q)|=O(10^{-6})$ considering values of p and q within the ranges of a typical artificial satellite, at zero inclination and in a circular orbit. The dominant part of H_0 corresponds to the Newtonian central field and here the term corresponding to the earth dynamical polar flattening has been included, so that the mean motion n is affected by an order of magnitude ε. The second term includes all other gravitational forces on the satellite resulting from the equatorial anomalies. An analytical solution of the original problem may be obtained by a succession of canonical transformations reducing the problem to one where only momenta are present in the mapped Hamiltonian (Brouwer, 1961). Each of these transformations are generated by functions assumed to have the same sequence of order of magnitude as the original Hamiltonian. In cases of resonance this is no longer true and one is forced to consider generating functions with a slower rate of approximation. This is the case of a satellite moving in an equatorial circular orbit. Here the critical argument is represented by the equatorial longitude of the satellite measured from an earth fixed reference frame. The dominant term in H_1 is given approximately by $H_1^*=(\mu/a)10^{-8}[A(p_1)\cos 2(\lambda-\theta)+B(p_1)\sin 2(\lambda-\theta)]$ where θ is the Greenwich hour angle at time t and all other parameters have been computed and incorporated to the numerical factor except for the ratio of the gravitational constant μ and the orbital mean radius a. Other terms with the same argument are to be included if one wishes to have a more precise result for the satellite position. This can actually be done by including a number of these terms and reducing the problem to the ideal resonance form.

The explicit form of the Hamiltonian, after all other angular variables have been eliminated, is (Kaula, 1966) given by

$$H(p,q)=f(p_1)+\omega_e p_2+\sum_{k\geq 2}[A_k(p_1)\cos k(q_1+q_2)+B_k(p_1)\sin k(q_1+q_2)] \tag{42}$$

The reference frequencies are given by $\omega_{10}=df(p_1)/dp_1=-n$, the mean angular motion in longitude, and $\omega_{20}=\omega_e$, the axial rotation of the Earth. For a near 24h satellite, one has a small value of the difference between these two frequencies and it is assumed a certain small parameter ε such that the difference between these frequencies is $O(\varepsilon^{1/2})$. This order is assumed in order to match with the order of magnitude of H_1 which is 10^{-6}, and considered O(ε). To this effect, one may think of ε as being of the order of 10^{-6}, so that the difference in those frequencies is at most 10^{-3}n.

By transforming the canonical variables to the new set defined by

$$y_1=q_1,\ y_2=q_1+q_2,\ p_1=x_1+x_2,\ p_2=x_2 \tag{43}$$

the Hamiltonian of the problem is mapped into

$$H(x,y) = f(x_1 + x_2) + \omega_e(x_2) + \sum_{k\geq 2}[A_k(x_1 + x_2)\cos ky_2 + B_k(x_1 + x_2)\sin ky_2] \tag{44}$$

The frequencies corresponding to the new variables are given by

$$\Omega_{10} = f'(x_1) = -n, \Omega_{20} = f'(x_2) + \omega_e = -n + \omega_e = O(\varepsilon^{1/2}) \tag{45}$$

It is seen that the frequency corresponding to the new variable x_2 is small. At exact resonance, it becomes zero, and the reference solution is stationary in the angular variable y_2. The problem is reduced to the normal form, represented by the Hamiltonian

$$H(x,y) = A_0(x) + \sum_{j\geq 2}[A_j(x)\cos jy + B_j(x)\sin jy] \tag{46}$$

where the only variables left are $x = x_2$ and $y = y_2$, while x_1 is constant and y_1 is easily obtained after the solution for x_2 and y_2 has been found. The order of magnitudes are ε for A_0, $\varepsilon^{1/2}$ for A_{0x}, and at least ε for all coefficients of the cosine and sine series. In the geostationary problem, one may define $A_0(x) = \mu^2/2(x_1 + x_2)^2 + \omega_e x_2 + O(\varepsilon)$ and the first few dominant terms are $A_2 = 3 C_{22} r^2 p_1^6$, $A_3 = (C_{33} + C_{31}) (3/8) r^3 p_1^8$, $A_4 = (C_{42} + C_{44}) 15 r^4 p_1^{10}$, and similar expressions for the B coefficients, where r is the ratio between the mean equatorial radius of the Earth and the gravitational constant μ for the Earth and C_{lm} and B_{lm} are the tesseral harmonic coefficients of the Earth potential field, measuring the dynamical equatorial distortion..

By means of a canonical transformation to new variables u,v defined by a generating function

$$\begin{aligned} S(u,y) &= uy + S_{1/2}(u,y) + S_1(u,y) + S_{3/2}(u,y) + \cdots \\ x = S_y &= u + S_{1/2,y} + S_{1,y} + S_{3/2,y} + \cdots \\ v = S_u &= y + S_{1/2,u} + S_{1,y} + S_{3/2,u} + \cdots \end{aligned} \tag{47}$$

the above Hamiltonian is mapped into the new form

$$H[x(u,v), y(u,v)] = K(u,v) = P(u) + Q(u)\cos 2v + R(u)\sin 2v \tag{48}$$

where the coefficients P and Q are to be computed by successive approximations as series of increasing order in ε. It is found that

$$P_0(u) = A_0(u), P_{1/2}(u) = 0, Q_0(u) = 0, Q_{1/2}(u) = 0, R_0(u) = 0, R_{1/2}(u) = 0 \tag{49}$$

while the $O(\varepsilon)$ and $O(\varepsilon^{3/2})$ equations read

$$\begin{aligned} &A_{0,u}S_{1/2,y} + (1/2)A_{0,uu}S^2_{1/2,y} + H_1 = P_1 + Q_1\cos 2y + R_1\sin 2y \\ &A_{0,u}S_{1,y} + H_{1,u}S_{1/2,y} = P_{3/2} - (4\sin 2y)S_{1/2,u}Q_1 + Q_{3/2}\cos 2y + (4\cos 2y)S_{1/2,u}R_1 \end{aligned} \tag{50}$$

By defining $Q_1 = A_2$ and $R_1 = B_2$, the equation for $S_{1/2,y}$ is given by

$$S_{1/2,y} = -\frac{A_{0,u}}{2A_{0,uu}} + \left\{\left(\frac{A_{0,u}}{2A_{0,uu}}\right)^2 + \frac{1}{A_{0,uu}}\left[\begin{array}{l} A_2(1+\cos 2y) + B_2(1+\sin 2y) - \\ -\sum_{j\geq 2} A_j\cos jy - \sum_{j\geq 2} B_j \sin jy \end{array}\right]\right\}^{1/2} = u - x \tag{51}$$

where the + sign has been chosen, representing one branch of the solution. This may be written as

$$S_{1/2,y} = -\frac{A_{0,u}}{2A_{0,uu}} + \sqrt{\Delta} = u - x \tag{52}$$

Clearly the sign of Δ will indicate whether libration or circulation occur, at given values of u and v(y) in the (v, u) phase plane. The problem corresponding to the Hamiltonian in the final form K(v, u) = P(u) + S(u) cos (2v-2α), has been discussed in details by Garfinkel (op. cit.). Under the above assumptions,

$$\tan 2\alpha = (B2 + B3 + \ldots)/(A2 + A3 + \ldots) = (B2/A2)(1+ B3/B2 - A3/A2+ \ldots) = (S_{22} / C_{22}) (1 + O(\varepsilon^{1/2})) \tag{53}$$

Introducing numerical values for S_{22} and C_{22} it is found that $\alpha = -29{,}85^{o}$. The Earth referred longitudes defined by $\alpha \pm \pi/2$ (60,15 degrees west and 119,85 degrees east of Greenwich), are the longitudes about which libration will occur. At exact resonance ($n = \omega_e$) these are stable equilibrium points for the geostationary satellite. A more precise calculation of these longitudes will show that the difference from the above values is just a few minutes of arc.

3. References

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