BUCKLING ANALYSIS OF SANDWICH PLATES: A THREE LAYER QUASI 3D FINITE ELEMENT MODEL

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Abstract:
A three-layer quasi 3D finite element model for buckling analysis of sandwich plates with laminated composite face-sheets is evaluated. In the model, the face-sheets are represented as Reissner-Mindlin plates and the core is modeled as a three-dimensional continuum. This representation allows accurate modeling for a wide range of core types. The three-dimensional problem is reduced to two dimensions by analytical through-thickness integration of the energy expressions for the evaluation of stiffness matrices. The stress stiffening effect is included in the quasi 3D model through the formulation of appropriate geometric stiffness matrices.
Linearized buckling analyses of sandwich plates are performed. The purpose is to compute critical loads and respective buckling modes from the solution of the eigenvalue problem by the subspace iteration method. A membrane state of stress can be easily established when applying the Reissner-Mindlin theory to the face-sheets. However, the quasi 3D formulation considers that the core behaves like a 3D structure connected to the face-sheets. This situation indicates the use of a general formulation that takes into account not only the membrane strains, but also all strain components to yield the geometric stiffness matrix.
Buckling problems are studied in order to validate the implemented code and demonstrate that linearized buckling loads of a sandwich plate may be strongly influenced by the variation of the core stiffness. The problem of buckling of sandwich plates with core discontinuity is also addressed.

Keywords: *Sandwich, composite, finite element, plate, and buckling.*

1. Introduction

Sandwich plates made using a honeycomb material for the core and metallic or laminated composite materials for the face-sheets are usually analyzed using simplified theories; good load predictions are typically obtained. The recent use of soft core materials such as PVC causes additional difficulties in the analysis because of the large differences between face-sheet and core stiffness. The structural analysis of sandwich plates with soft cores, therefore, requires a more sophisticated theory than classical sandwich theories. When analyzing this type of sandwich plate, 3D analyses are often necessary because of the required solution accuracy. However, thin face-sheets and thick cores place severe restrictions on the use of solid finite elements because of three-dimensional element aspect ratios and associated numerical errors. Although the idea of designing a strong, durable and lightweight structure is widely accepted, the design tools are not well established when a sandwich structure has a very flexible core.

The study of the buckling behavior for the sandwich structures using beam and plate formulations has been the aim of many researchers. Cook (1975) formulated a finite element analysis for sandwich plates. The model is based on quadrilateral elements with 12 degrees of freedom; this yields significant errors when appreciable transverse shear deformation is present. Smith (1986) demonstrated the folded plate method for multilayer laminate or sandwich panels. These approaches require considerable insight with respect to the inclusion of the important terms in the formulation and may yield difficulties when the core is modeled as a three dimensional continuum. Cook (1981) developed a general formulation, where all strain components include the full non-linear terms. The higher-order terms yield the non-linear strain energy, from where the general stress stiffness matrix may be derived.

Most effort has been directed to the study of the linearized buckling problem (for example, Pagano, 1970; Cheung et al., 1982; Rao, 1985). Hunt et al. (1988) pointed out the tendency to more catastrophic and complex forms of failure for sandwich structures with a flexible core. They modeled a sandwich plate that has a triggering bifurcation with an overall mode of buckling and it is rapidly followed by a secondary bifurcation with an unstable combination of local modes or the wrinkling of one of the faces. Da Silva and Hunt (1990) extended this work studying the importance of core orthotropy to maximize the effect of destabilization of the sandwich structure.

Frostig and Baruch (1993) used a higher-order theory to analyze the buckling of sandwich beams with transversely flexible core. They formulated closed-form solutions for the non-linear buckling equations by means of perturbation techniques. Waszczyszyn et al. (1994) described a finite element procedure to assess the sensitivity of a linearized buckling analysis of sandwich plate to imperfections. Bathe (1996) compares this procedure to the complete incremental non-linear solution. Non-linear analysis estimates the maximum load that a sandwich plate can support prior to structural collapse. Numerically, an incremental analysis helps in the calculation of the plate response. However, the complete incremental non-linear solution of the plate up to collapse and beyond can be computationally expensive, and a linearized buckling analysis can be of value to estimate the lower buckling loads when the prebuckling displacements are small. Sokolinsky and Frostig (1999) analyze the effect of different types of boundary conditions on the critical load using a high-order non-linear formulation obtained by Frostig and Baruch (1993).

As an extension of a previous finite element formulation for the vibration analysis of sandwich plates (Nabarrete et al, 2001), the present research is intended to develop a three layer quasi 3D tool to analyze buckling problems. This design tool is intended to solve three-dimensional sandwich plate problems with the simplicity of a two-dimensional finite element model. In the element formulation, through-thickness integration of strain energy is completed analytically; this gives the appearance of a two-dimensional finite element procedure, and leads to computational efficiency. A Fortran code was generated and a comparison of results was made between this formulation and 3D finite element calculations. The presented results provide buckling loading predictions using a broad range of core stiffness. When large differences between face-sheets and core stiffness are present, the current model always provides good agreement with refined 3D finite element results. The present three layer quasi 3D model avoids numerical problems associated with 3D finite element aspect ratios and has the advantage of two-dimensional geometric modeling.

2. Mathematical Model

A linearized buckling analysis is typically performed with the calculation of the membrane state of stress, which describes the stress stiffening condition in the plate. In this work, the stress stiffening effect is included in the quasi 3D model through the formulation of appropriate geometric stiffness matrices. Then, a linearized buckling analysis is performed to predict the critical buckling loads. The face-sheets are formulated using the Reissner-Mindlin plate theory and a membrane state of stress is easy to establish. However, the core is modeled as a 3D continuum connected to the face-sheets and it is difficult to associate a membrane state of stress if arbitrary elastic properties are assumed for the core. This requires the use of a general formulation (Cook, 1981) that takes into account not only the membrane strains, but all strain components in the development of the geometric stiffness matrix.

The linearized buckling analysis is performed using a geometric stiffness matrix that is independent of elastic properties, but depends on the element geometry, displacement field and state of stress.

2.1 Kinematic equations

The face-sheets include transverse shear effects, which are consistent with a Reissner-Mindlin plate theory. Both face-sheets are modeled using bi-cubic trial functions of x and y to represent the in-plane displacement and curvature in the plane orthogonal directions (u and v) as well as the displacement in the transverse direction (w). Figure (1) illustrates the transverse coordinates used for the face-sheets and core.

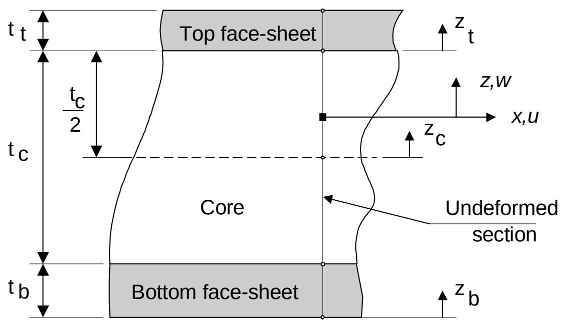

Figure 1 – Cross-section of the sandwich plate.

Sandwich structures may contain light metallic honeycomb as a core material, thus the in-plane normal and shear stiffness are small compared to the transverse stiffness (Frostig and Baruch, 1994). Some authors (Thomsen, 1993; Frostig and Baruch, 1996) had set the in-plane stiffness of the core to zero in their formulations. However, when flexible or rigid foams are used the core material is effective isotropic and therefore no simplification is possible. The present work follows the approach developed in Oskooei and Hansen (2000) that adopted cubic trial functions in z for the in-plane displacements u and v, and a quadratic trial function in z for the transverse displacement w; thus, the energy associated with in-plane normal and shear stiffness may be neglected if appropriate. The use of these trial functions will not induce artificial stiffening. When using foam cores, the difference between in-plane normal and shear stiffness does not appear and the strain energy associated with them may be computed as usual. Foam cores have low values for transverse stiffness in comparison with the face-sheets and the current formulation is capable of yielding accurate results for this situation.

The in-plane bi-cubic trial functions used to represent the displacements are:

Bottom face:
$$\begin{aligned} u(x,y,z) &= a_0(x,y) + a_1(x,y)\, z, \\ v(x,y,z) &= b_0(x,y) + b_1(x,y)\, z, \\ w(x,y,z) &= l_0(x,y), \end{aligned} \tag{1}$$

Core:
$$\begin{aligned} u(x,y,z) &= c_0(x,y) + c_1(x,y)\, z + c_2(x,y)\, z^2 + c_3(x,y)\, z^3, \\ v(x,y,z) &= d_0(x,y) + d_1(x,y)\, z + d_2(x,y)\, z^2 + d_3(x,y)\, z^3, \\ w(x,y,z) &= m_0(x,y) + m_1(x,y)\, z + m_2(x,y)\, z^2, \end{aligned} \tag{2}$$

Top face:
$$\begin{aligned} u(x,y,z) &= e_0(x,y) + e_1(x,y)\, z, \\ v(x,y,z) &= f_0(x,y) + f_1(x,y)\, z, \\ w(x,y,z) &= n_0(x,y), \end{aligned} \tag{3}$$

In above, u, v and w are displacements in the *x, y* and *z* directions respectively. Parameters a_0, a_1, b_0, b_1, c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3, e_0, e_1, f_0, f_1, l_0, m_0, m_1, m_2 and n_0 are bi-cubic functions of *x* and *y*. It is observed that both, top and bottom face-sheets, have a similar formulation. Subscripts *b*, *c* and *t* are used to identify formulations for the bottom face-sheet, core and top face-sheet respectively. As the bottom and top face-sheet have similar formulations, only the bottom face-sheet will be described in the following expressions.

The full non-linear strain-displacement relations have to be considered when stress-stiffening effects are included in the formulation (Cook, 1981). Therefore the strain is represented as the sum of linear and non-linear components in this work. The linear strain components are given for the face-sheets and core respectively by Eqs. (4) and (5). In these expressions, $\{\varepsilon_0\}$, $\{\kappa_i\}$ and $\{\Gamma\}$ are the in-plane strains, curvatures and out-of-plane shear strains vectors.

$$\left\{\varepsilon^L\right\}_b = \begin{bmatrix}\varepsilon_x & \varepsilon_y & \gamma_{xy} & \gamma_{zx} & \gamma_{yz}\end{bmatrix}_b^T = \begin{Bmatrix} \{\varepsilon_0\} + z\,\{\kappa_1\} \\ \{\Gamma\} \end{Bmatrix}_b \tag{4}$$

$$\left\{\varepsilon^L\right\}_c = \begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{yz} & \gamma_{zx} & \gamma_{xy}\end{bmatrix}_c^T = \{\varepsilon_0\} + z\,\{\kappa_1\} + z^2\{\kappa_2\} + z^3\{\kappa_3\} \tag{5}$$

2.2 Constitutive equations

The core is assumed to be orthotropic with the principal material directions rotated around the *z*-axis. Therefore, the constitutive relation for the core has the form of that for monoclinic materials (Jones, 1975).

The face-sheets may be formed as a composite laminate. The appropriate constitutive equation for the k^{th} lamina of the bottom face-sheet is represented by:

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{zx} \\ \tau_{yz}\end{Bmatrix}_k = \left[\begin{array}{ccc|cc} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ \hline 0 & 0 & 0 & \overline{Q}_{55} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{44} \end{array}\right]_k \begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{zx} \\ \gamma_{yz}\end{Bmatrix}_k = \begin{bmatrix} [\,\overline{Q}_M\,] & [0] \\ [0] & [\,\overline{Q}_S\,] \end{bmatrix}_k \begin{Bmatrix} \{\varepsilon_0 + z\kappa_1\} \\ \{\Gamma\} \end{Bmatrix}_k \tag{6}$$

2.3 Variational formulation

The solution of the buckling problem by means of a two-dimensional finite element formulation requires the minimization of a functional $\Pi = U - \Delta W$, where *U* is the sum of the face-sheet and core strain energies. ΔW is the change of the potential energy due to the applied forces.

Integrating the bottom face-sheet strain energy with respect to *z* and relating the strains to the p^{th} element nodal displacements, the strain energy for the bottom face-sheet becomes:

$$U_b = \frac{1}{2}\int_A \begin{Bmatrix}\{\varepsilon_0\} \\ \{\kappa_1\} \\ \{\Gamma\}\end{Bmatrix}_b^T \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [A^*] \end{bmatrix}_b \begin{Bmatrix}\{\varepsilon_0\} \\ \{\kappa_1\} \\ \{\Gamma\}\end{Bmatrix}_b dA = \frac{1}{2}\{\overline{u}\}_p \left[\iint_A [\mathrm{B}]_b^T\,[\mathrm{D}]_b\,[\mathrm{B}]_b\, dA\right]\{\overline{u}\}_p \tag{7}$$

where,

$$\left([A],[B],[D] \right)_b = \sum_{k=1}^{n_b} \left(\int_{z_{k-1}}^{z_k} \left[\overline{Q}_M \right]_k \left(1, z, z^2\right) dz \right) \tag{8}$$

$$\left(\left[A^*\right] \right)_b = \sum_{k=1}^{n_b} \left(\int_{z_{k-1}}^{z_k} \left[\overline{Q}_S \right]_k \, dz \right) \tag{9}$$

Integrating the core strain energy with respect to z and relating the strains to the p^{th} element nodal displacements, the core strain energy takes the form

$$U_c = \frac{1}{2} \int_A \begin{Bmatrix} \{\varepsilon_0\} \\ \{\kappa_1\} \\ \{\kappa_2\} \\ \{\kappa_3\} \end{Bmatrix}_c^T \begin{bmatrix} [A] & [B] & [D] & [E] \\ [B] & [D] & [E] & [F] \\ [D] & [E] & [F] & [G] \\ [E] & [F] & [G] & [H] \end{bmatrix}_c \begin{Bmatrix} \{\varepsilon_0\} \\ \{\kappa_1\} \\ \{\kappa_2\} \\ \{\kappa_3\} \end{Bmatrix}_c dA = \frac{1}{2} \{\overline{u}\}_p \left[\int_A [\mathsf{B}]_c^T \, [\mathsf{D}]_c \, [\mathsf{B}]_c \, dA \right] \{\overline{u}\}_p \tag{10}$$

Several matrices compose the stiffness matrix and are evaluated by:

$$\left([A],[B],[D],[E],[F],[G],[H] \right)_c = \int_{-z_c}^{z_c} \left[\overline{C} \right]_c \left(1, z, z^2, z^3, z^4, z^5, z^6\right) dz \tag{11}$$

where, $[\,\overline{C}\,]_c$ is the elasticity matrix for the core.

For the bottom face-sheet, using a perturbation technique ΔW can be shown to be (Dawe and Roufaeil, 1982):

$$\begin{aligned} \Delta W_b = & -\frac{1}{2} \int_A \int_z \left[\sigma_x^0 (u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \right]_b dz\, dA - \frac{1}{2} \int_A \int_z \left[\sigma_y^0 (u_{,y}^2 + v_{,y}^2 + w_{,y}^2) \right]_b dz\, dA - \\ & - \int_A \int_z \left[\tau_{xy}^0 \left(u_{,x}\, u_{,y} + v_{,x}\, v_{,y} + w_{,x}\, w_{,y} \right) \right]_b dz\, dA \end{aligned} \tag{12}$$

where, σ_x^0, σ_y^0 and τ_{xy}^0 corresponds to the existing stress caused by the loading in a prebuckling state.

Dawe and Roufaeil (1982) considered the buckling of rectangular Mindlin plates, where the inplane higher order strain terms were not considered because they do not contribute to the plate bending. Also, the derivatives of the curvature with respect to x and y have been considered due to their significance for the buckling of moderately thick plates, which is akin to the significance of rotary inertia in the free vibration problem. Equation (12) is rewritten as

$$\Delta W_b = -\frac{1}{2} \int_A \int_z \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ l_{0,x} \\ l_{0,y} \\ 0 \end{matrix} \right\}^T [S_0]_b \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ l_{0,x} \\ l_{0,y} \\ 0 \end{matrix} \right\} + 2z \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ l_{0,x} \\ l_{0,y} \\ 0 \end{matrix} \right\}^T [S_0]_b \left\{ \begin{matrix} a_{1,x} \\ a_{1,y} \\ 0 \\ b_{1,x} \\ b_{1,y} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\} + z^2 \left\{ \begin{matrix} a_{1,x} \\ a_{1,y} \\ 0 \\ b_{1,x} \\ b_{1,y} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\}^T [S_0]_b \left\{ \begin{matrix} a_{1,x} \\ a_{1,y} \\ 0 \\ b_{1,x} \\ b_{1,y} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\} dz\, dA. \tag{13}$$

where,

$$[S_0]_b = \begin{bmatrix} [S_b] & 0 & 0 \\ 0 & [S_b] & 0 \\ 0 & 0 & [S_b] \end{bmatrix} \tag{14}$$

The square matrix $[S_b]$ represents the existing stresses for the bottom face-sheet and is given by

$$[S_b] = \begin{bmatrix} \sigma_x^0 & \tau_{xy}^0 & 0 \\ \tau_{xy}^0 & \sigma_y^0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_b \tag{15}$$

Both curvature and membrane strain are assumed to be representative terms for the face sheets in the buckling process because the in-plane loading is applied to the mid-plane of the face-sheets while the reference plane is at the middle plane of the core. Therefore, the following expression is obtained after integrating Eq. (13) through the bottom face-sheet thickness.

$$\Delta W_b = -\frac{1}{2}\{\bar{u}\}_p^T \left[\int_A \begin{bmatrix} [G_w] \\ [G_\psi] \end{bmatrix}_b^T \begin{bmatrix} [N^0] & [M^0] \\ [M^0] & [L^0] \end{bmatrix}_b \begin{bmatrix} [G_w] \\ [G_\psi] \end{bmatrix}_b dA \right] \{\bar{u}\}_p \tag{16}$$

where, $[G_w]$ and $[G_\psi]$ are transformation matrices and relate trial function derivatives to through-thickness nodal displacements. $[N^0]$, $[M^0]$ and $[L^0]$ result from integration of both, curvature and membrane strain. These are assumed as representative terms for the face sheets in the buckling process of sandwich plates because the in-plane loading is applied to the mid-plane of the face-sheets while the reference plane is at the middle plane of the core.

$$\left([N^0], [M^0], [L^0] \right)_b = \int_{z_1}^{z_2} [S_0]_b \left(1, z, z^2 \right) dz \tag{17}$$

The three dimensional nature of the core indicates the use of a general formulation (Cook, 1981) that incorporates all non-linear strain components in the geometric stiffness matrix. The change in the potential energy for the core yields

$$\Delta W_c = -\iint_{A\,z} \left(\sigma_x^0 \varepsilon_x^N + \sigma_y^0 \varepsilon_y^N + \sigma_z^0 \varepsilon_z^N + \tau_{xy}^0 \gamma_{xy}^N + \tau_{yz}^0 \gamma_{yz}^N + \tau_{zx}^0 \gamma_{zx}^N \right)_c dz\, dA \tag{18}$$

Defining $[G]_c$ as a transformation matrix that relates trial function derivatives to the p^{th} element nodal displacements, the change in the potential energy can be rewritten as

$$\Delta W_c = -\frac{1}{2}\{\bar{u}\}_p^T \left\{ \int_A \left[\int_z [G]_c^T \begin{bmatrix} [S_c] & 0 & 0 \\ 0 & [S_c] & 0 \\ 0 & 0 & [S_c] \end{bmatrix} [G]_c \, dz \right] dA \right\} \{\bar{u}\}_p \tag{19}$$

where, the square matrix $[S_c]$ represents the existing stresses for the core and is given by

$$[S_c] = \begin{bmatrix} \sigma_x^0 & \tau_{xy}^0 & \tau_{zx}^0 \\ \tau_{xy}^0 & \sigma_y^0 & \tau_{yz}^0 \\ \tau_{zx}^0 & \tau_{yz}^0 & \sigma_z^0 \end{bmatrix}_c \tag{20}$$

2.4 Isoparametric formulation

The assembly of the face-sheets and core geometric stiffness matrices is easier if the polynomial coefficients of the trial functions are expressed in terms of through-thickness nodal displacements. This transformation to through-thickness nodal displacements is performed as shown in Fig. (2). The three layer quasi 3D model have a total of 15 through-thickness nodal variables, namely, the coefficients u_1, u_2, u_3, u_5, u_6, u_7, v_1, v_2, v_3, v_5, v_6, v_7, w_2, w_4, and w_6. With a 16-node isoparametric in-plane interpolation scheme there are 240 degrees of freedom for the element.

In the integration process of Eq. (7) the element nodal degrees-of-freedom vector $\{\bar{u}\}_p$ is considered as constant and the strain energy for the bottom face-sheet is written as

$$U_b = \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T \left[\iint [\mathrm{B}]_b^T [\mathrm{D}]_b [\mathrm{B}]_b \,|J|\, d\xi\, d\eta \right]\{\bar{u}\}_p = \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T\,[K_b]_p\,\{\bar{u}\}_p \tag{21}$$

Similar expressions to Eq. (21) are derived for the top face-sheet and core and the strain energy of the sandwich plate is then obtained for *n* discrete finite elements as:

$$U = \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T[K]_p\{\bar{u}\}_p = \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T[K_b]_p\{\bar{u}\}_p + \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T[K_c]_p\{\bar{u}\}_p + \frac{1}{2}\sum_{p=1}^{n}\{\bar{u}\}_p^T[K_t]_p\{\bar{u}\}_p \tag{22}$$

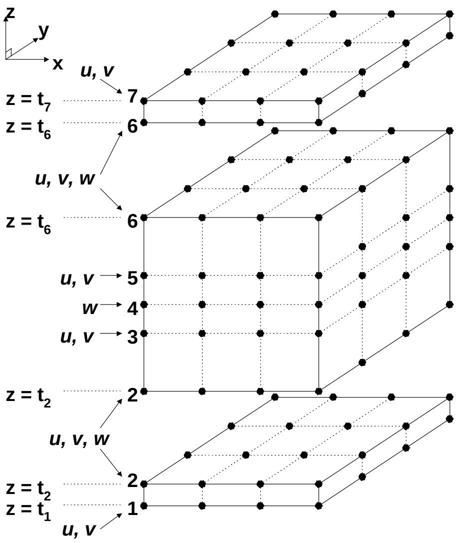

Figure 2 – Schematic of quasi 3D element (through-thickness degrees of freedom).

By adopting isoparametric local variables in Eq. (16), the geometric stiffness matrix for the bottom face-sheet becomes

$$[K_\sigma]_b = \iint \begin{bmatrix} [G_w] \\ [G_\psi] \end{bmatrix}_b^T \begin{bmatrix} [N^0] & [M^0] \\ [M^0] & [L^0] \end{bmatrix}_b \begin{bmatrix} [G_w] \\ [G_\psi] \end{bmatrix}_b |J|\, d\xi\, d\eta \tag{23}$$

Similarly, the geometric stiffness matrix for the core is

$$[K_\sigma]_c = \iint \left[\int_z [G]_c^T \begin{bmatrix} [S_c] & 0 & 0 \\ 0 & [S_c] & 0 \\ 0 & 0 & [S_c] \end{bmatrix} [G]_c\, dz \right] |J|\, d\xi\, d\eta \tag{24}$$

Computing the sum of the change in the potential energy for the bottom and top face-sheets and core yields:

$$\Delta W = -\frac{1}{2}\left(\{\bar{u}\}_p^T[K_\sigma]\{\bar{u}\}_p\right) = -\frac{1}{2}\left(\{\bar{u}\}_p^T[K_\sigma]_b\{\bar{u}\}_p + \{\bar{u}\}_p^T[K_\sigma]_c\{\bar{u}\}_p + \{\bar{u}\}_p^T[K_\sigma]_t\{\bar{u}\}_p\right) \tag{25}$$

2.5 Linearized buckling formulation

The value of the critical buckling load can be extracted by the solution of the eigenvalue problem expressed by:

$$[\,[K] + \lambda\,[K_\sigma]\,]\{\Delta\} = \{0\} \tag{26}$$

The eigenvalue λ represents the critical buckling load for the sandwich plate. The vector of displacements $\{\Delta\}$ (eigenvector) characterizes the buckled shape for the respective critical buckling load, but not its magnitude.

The subspace iteration method (Bathe, 1996) is used to determine the buckling loads and respective mode shapes. The hypotheses for the linearized buckling of sandwich plates are: forces applied to the sandwich are fixed in magnitude, global direction and point of application; buckling displacements and rotations are small; forces and stresses remain essentially constant during the buckling displacement, finally, the problem is linear for the displacement variables.

3. Numerical results for the linearized buckling analysis

Two examples are analyzed in order to demonstrate the efficiency of the developed quasi 3D model to obtain buckling loads and respective buckling modes. Initially, a uniform sandwich plate is analyzed and the effect of the variation of the core stiffness on the sandwich plate buckling load is assessed. A second example deals with the linearized buckling analysis of sandwich plates with a discontinuous core. This type of defect is of practical significance and lowers the critical buckling load.

3.1 Influence of core stiffness on the buckling load

In this example, a sandwich plate is analyzed to evaluate the critical buckling load as a function of the core stiffness. The material properties are described in Tab. (1). As depicted in Fig. (3), the plate is clamped at both face-sheets along one of the shorter edges whereas a uniform distributed compressive longitudinal load is applied to both face-sheets at the opposing edge in order to maintain the load symmetry for the model. It should be observed that the existence of any asymmetry in the load causes the plate to bend and, as a consequence, no bifurcation buckling occurs. The two other edges are unrestrained resulting in a cantilever plate.

Table 1 – Material properties for the cantilever sandwich plate.

Face Sheets	Modulus of elasticity (E)	68258 MPa
	Poisson ratio (ν)	0.33
Core	Modulus of elasticity (E)	1000 MPa
	Poisson ratio (ν)	0.3

The results are compared to those obtained from a Nastran finite element model using solid elements. The quasi 3D model was built with 16 elements and 175 nodes with a total of 2625 degrees of freedom. The Nastran model was built with 576 quadratic 20-node solid elements with a total of 3165 nodes and 9495 degrees of freedom.

Table (2) presents the first three buckling loads of the cantilever sandwich plate as a function of core stiffness. The second and third buckling loads are reported in the table to demonstrate the accuracy of the quasi 3D model for higher modes; this accuracy might be useful, for example, for optimization purposes.

Table 2 – Comparison of longitudinal buckling loads for the cantilever sandwich plate.

E_c (MPa)	1st mode		2nd mode		3rd mode	
	Quasi 3D	3D solid (Nastran)	Quasi 3D	3D solid (Nastran)	Quasi 3D	3D solid (Nastran)
10000	-7990.0	-8014.8	-70026.0	-71129.3	-166100.0	-172476.8
3000	-7174.3	-7200.2	-59687.0	-60768.7	-123050.0	-127869.5
1000	-6818.0	-6846.6	-49855.0	-50754.6	-100580.0	-103989.7
301	-6327.7	-6358.7	-33402.0	-33984.1	-50546.0	-52182.5
151	-5809.9	-5842.0	-22901.0	-23308.3	-30001.0	-30896.8
30	-3535.6	-3563.7	-6652.9	-6760.1	-7588.2	-7711.0

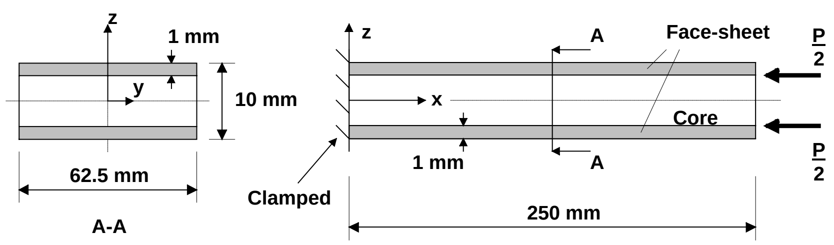

Figure 3 – Sandwich plate submitted to longitudinal in-plane compression.

Figure (4) represents the three-dimensional view of the first buckling mode of the cantilever sandwich plate. The variation of the critical buckling loads as a function of the core transverse stiffness is depicted in the Fig. (5). The plots compare the quasi 3D model and the Nastran code results; it can be observed that the two solutions are in close agreement with no significant differences between them. As expected, the resulting curves indicate that the critical buckling load increases as the core stiffness increases within the range of core stiffness studied. When the core transverse stiffness is low in comparison to the face-sheets, lower shear stresses result in the core and high relative displacements occur between the face-sheets causing the reduction of the critical buckling loads. This example demonstrates that the code is robust and useful when calculating the geometric stiffness matrix for the buckling analysis of sandwich plates made with either a rigid or flexible core.

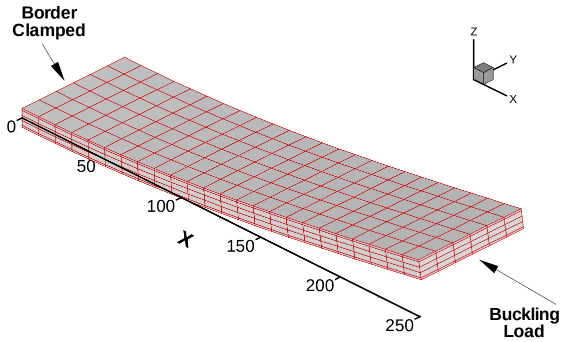

Figure 4 – Sandwich plate buckled with longitudinal in-plane compression.

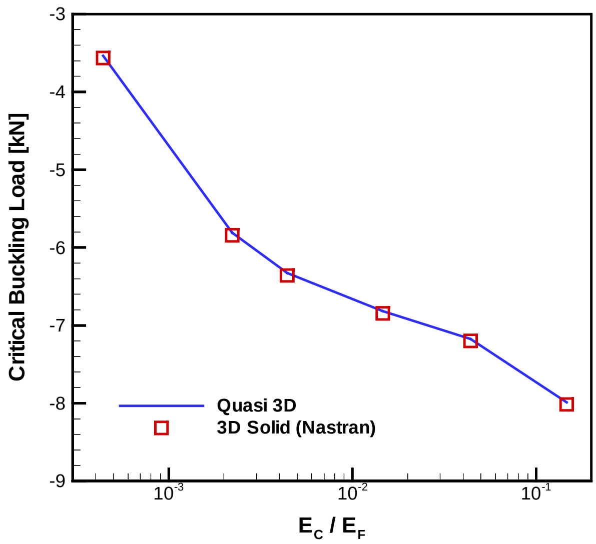

Figure 5 – Buckling load comparison for changing core stiffness.

3.2 Buckling of sandwich plates with discontinuous core

Allen (1998) studied the effect of discontinuities in the core on the buckling load of sandwich plates. This is a typical defect in sandwich plates that occurs in the core when it is made from contiguous blocks of material and the foam core is provided in standard dimensions that are smaller than the necessary. In this case, blocks are placed contiguously to form the core and a small gap may exist between them. Figure (6) represents this situation.

A buckling analysis of sandwich plates with discontinuous core was performed to demonstrate the ability of the quasi 3D model to model this type of defect. The dimensions of the sandwich plate and the core gap are described in Tab. (3), and the material properties of the sandwich plate are those described in Tab. (1). Figures (7) and (8) show the results for two buckling modes of a sandwich plate with discontinuous core; both a coarse and refined mesh are used in the region of the discontinuity to assess the accuracy of the Quasi 3D model results. The values of critical buckling loads are presented in the figures and are quite close for both meshes demonstrating the convergence of the formulation. The coarse and refined meshes for the gap region of the core use 2 and 6 elements, respectively. The quasi 3D model totaled 22 elements with 238 nodes and 3570 degrees of freedom for the refined mesh analysis. As expected, the refined mesh result for the critical buckling loads has a lower value and is more accurate for the local buckling in the core gap region.

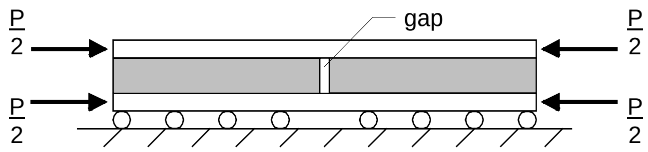

Figure 6 – Sandwich plate with discontinuous core.

The quasi 3D finite element is modeled with the simplicity typical of a two-dimensional model; only the *x* and *y* coordinates are needed to define the element boundary. The input data for the core discontinuity modeling is simply implemented by adopting a very low stiffness in the material property data for the elements in the gap region.

Table 3 – Dimensions of sandwich plate with discontinuous core.

Plate Dimension		Thickness	
Length (*l*)	250 mm	Bottom face-sheet	0.5 mm
Width (*w*)	62.5 mm	Top face-sheet	0.5 mm
Gap length (*d*)	6.0 mm	Core	9.0 mm

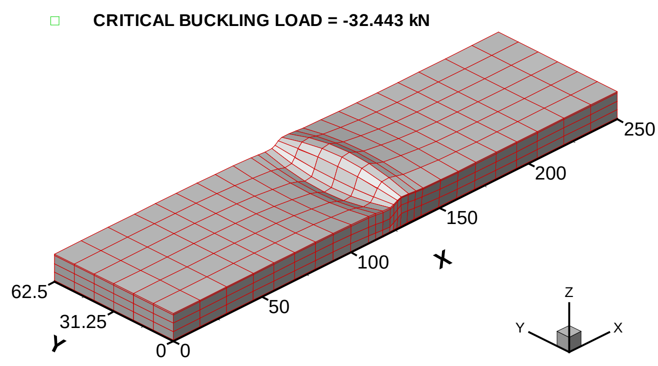

Figure 7 – Buckling of sandwich plate with discontinuous core.

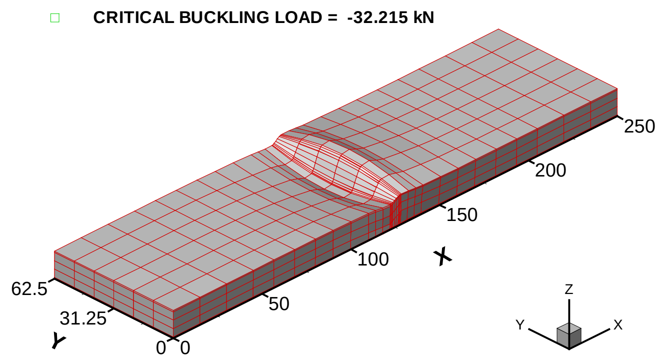

Figure 8 – Buckling of sandwich plate with discontinuous core (refined mesh).

4. Conclusions

Geometric stiffness matrices for the face-sheets and core have been derived and incorporated into the three layer quasi 3D model. Reissner-Mindlin plate theory has been used for the face-sheets considering the transverse shear strains and respective stresses. A general formulation (Cook, 1981) has been adapted for the core taking into account the strains in all directions as result of a solid behavior. This formulation can be adapted to honeycomb cores, which have higher stiffness in transverse direction versus an insignificant stiffness in the in-plane directions. It can also be applied for flexible foams that have low stiffness in all directions in comparison with the face-sheets. The global geometric stiffness matrix was assembled for the sandwich plate and a linearized buckling analysis has been performed to extract the buckling loads of the plate for three different configurations.

The validation of the numerically calculated buckling loads was obtained by comparing the three layer quasi 3D model with a solid finite element model using Nastran. The critical buckling load of a cantilever sandwich plate with longitudinal in-plane distributed load was computed as function of core stiffness. The comparison of both formulations showed that using the three layer quasi 3D model, good estimates were obtained for the buckling loads and respective buckling shapes. The present model leads to accurate solutions even when large differences between core and face-sheets transverse stiffness exists.

Buckling of sandwich plates with core discontinuity were also investigated. The results indicate that the three layer quasi 3D model yields accurate solutions with great modeling facilities. A convergence study was performed in both examples to validate the formulation and demonstrate the accuracy of the model.

When thin face-sheets are modeled in a sandwich structure, the current quasi 3D model may have some distinct computational advantages to solve the linearized buckling analyses in comparison to an usual 3D finite element model. Also, it should be concluded that through-thickness plate geometry does not affect the number of degrees of freedom of a quasi 3D model; however, if the face-sheets are thin, a 3D solid model require an excessive number of through-thickness elements to overcome numerical difficulties resulting from poor element aspect ratio. The current calculations illustrate that the quasi 3D model obtains accurate solutions even with a small number of elements.

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