DEVELOPMENT OF SOFTWARE FOR SYNTHESIS OF TRUSS-BASED COMPLIANT MECHANISMS USING TOPOLOGY OPTIMIZATION

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Abstract. *This paper presents a method for the synthesis of truss-based bidimensional compliant mechanisms using Topology Optimization. In compliant mechanisms, elastic deformation is the source of motion, instead of joints. This fact introduces advantages, such as a smaller number of parts, less wear and backlash. Thus, the use of this class of mechanisms is important in precision mechanics, biomedical applications and in micro devices, called "MEMS", in which there are difficulties related to micro assembly. The Topology Optimization method combines a finite element algorithm with an optimization algorithm, based on the Sequencial Linear Programing (SLP). The use of trusses increases computational efficiency, since the numerical process in this case is faster than for a continuum domain. Area penalization is implemented in the software, as well as a procedure for the sysnthesis of multi-flexible mechanisms, in discrete domains. The results show that the use of area penalization is an efficient way to generate better defined topologies. A multi-flexible mechanism was synthetized and simulated considering two design requirements. The simulation confirms that design requirements were satisfied.*

Keywords. *Topology Optimization, Compliant Mechanisms, Trusses, Microelectromechanisms, MEMS.*

1. Introdution

Compliant mechanisms are those which use the flexibility of their constituent elements as the source of motion. This fact distinguishes them from traditional mechanisms, designed as rigid-body links with movement freedom only at the joints (Kota et al., 1999; Frecker et al. 1996). Compliant mechanisms present advantages in relation to rigid-body mechanisms: fewer components, easier manufacturing, less wear, backlash and friction. Besides, they do not need lubrication and have built-in restoring force (Sigmund, 1996). This class of mechanisms have large application in biomedical engineering, precision mechanics and, more recently, in "MEMS", structures with dimensions changing from hundreds of micrometers to few millimeters, that combine mechanical microcomponents, forming systems.

Due to the complex elastic behavior of compliant mechanisms, trial and error methods have been many times used in their development. However, they depend on the designer's physical intuition and become inefficient as increases the number of design variables. Therefore, systematic methods have been developed to design compliant mechanisms. There are two basic approaches to the problem: kinematic synthesis and continuum synthesis. Examples for the first type are chain methods. They consist in generating rigid-body mechanisms and gradually introducing flexibility in some points, creating concentrated flexibility mechanisms (Her et al., 1987). The continuum synthesis (Anathasuresh et al., 1994) tries to synthetize distributed flexibility mechanisms, using the Topology Optimization method (Bendsøe et al., 1988). This method, originally intended to design minimum weight and maximum stiffness structures (Suzuki et al., 1991), has been adapted to kinematic and structural requirements of compliant mechanisms (Ananthasuresh et al., 1994; Sigmund, 1995; Frecker, et al., 1996; Nishiwaki et al., 1998). For continuum domains, Topology Optimization can be implemented using a material model called SIMP ("Simple Isotropic Material with Penalization") (Bendsøe, 1989; Zhou et al., 1991; Mlejnek, 1992). A methodology for the synthesis of multi-flexible mechanisms was also presented for continuum domains (Nishiwaki et al., 2001).

The objective of this work is the development of a method for the synthesis of truss based compliant mechanisms using Topology Optimization that incorporates penalization of areas and multi-flexible mechanisms design. The advantage of working with trusses is that the computational efficiency is increased, since the numerical process in this case is faster than for a continuum domain. The use of penalization and multi-flexibility mechanisms design (Nishiwaki et al., 2001) are implemented in this work for discrete domains.

This paper is organized as follows: in section 2.1 a formulation for the synthesis of compliant mechanisms considering kinematic and structural requirements is presented. Section 2.2 describes the use of area penalization. The multi-flexibility formulation is presented in section 2.3. In section 4, examples of mechanisms generated by the software implemented in this work are presented and the effect of area penalization and the concept of multi-flexibility are discussed. In section 5 some conclusions are given and a future work is proposed.

2. Problem formulation

2.1. Formulation for the synthesis of truss based compliant mechanisms

Before introducing the concepts of area penalization and multi-flexibility, the design of compliant mechanisms with single flexibility is briefly presented (Frecker et al., 1996; Nishiwaki et al., 2001). The design of compliant mechanisms must satisfy kinematic and structural requirements. Kinematic requirements consists of maximizing the deflection at some point of interest, when a particular load is applied to the structure. Structural requirements are related to the maximization of stiffness when the load points are fixed and the mecanism is subjected to the reaction force of the body in contact with it.

A generic design problem is showed in Fig. (1a). Applying force $\mathbf{f}_A$ to point A, displacement $\boldsymbol{\Delta}$ is desired at point B. The first part of the optimization problem corresponds to the maximization of the displacement $\boldsymbol{\Delta}$, that has the same direction of the dummy load $\mathbf{f}_B$, applied at point B (Fig. (1b)).

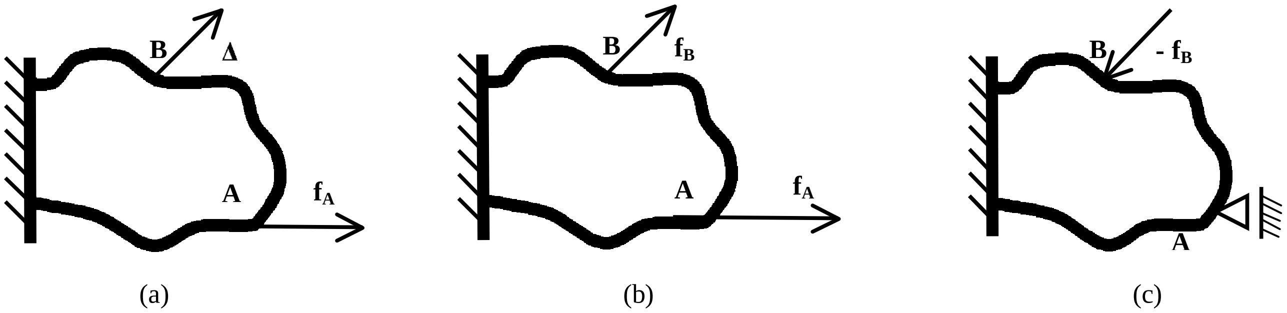


Figure 1. (a) Problem statement; (b) Kinematic requirement; (c) Structural requirement.

The displacement field caused by the application of $\mathbf{f}_A$ is $\mathbf{u}_A$, and $\mathbf{v}_B$ is the displacement caused by $\mathbf{f}_B$. Considering the force $\mathbf{f}_B$ to be unitary, $\mathbf{f}_B^T\mathbf{u}_A$ represents the magnitude of the displacement $\boldsymbol{\Delta}$. However, using Betti's Reciprocal Theorem:

$$\mathbf{f}_B{}^T\mathbf{u}_A = \mathbf{f}_A{}^T\mathbf{v}_B = L_2 \tag{1}$$

The term L_2 is called the mutual energy. Finally, the kinematical problem can be posed as the maximization of the mutual energy $\mathbf{f}_A{}^T\mathbf{v}_B$. Using equilibrium equation, $\mathbf{f}_A{}^T$ can be substituted by $\mathbf{u}_A{}^T\mathbf{K}_1$, where $\mathbf{K}_1$ is the global stiffness matrix ($\mathbf{K}_1 = \mathbf{K}_1{}^T$):

$$\mathbf{K}_1\mathbf{u}_A = \mathbf{f}_A \Rightarrow \mathbf{u}_A{}^T\mathbf{K}_1 = \mathbf{f}_A{}^T \tag{2}$$

Thus, the first part of the optimization problem is the maximization of the following objective function:

$$L_2 = (\mathbf{u}_\mathbf{A}{}^T\mathbf{K}_1\mathbf{v}_\mathbf{B}) \tag{3}$$

subjected to the constraints in Eq. (5)

If only the kinematic requirement is satisfied, a weak structure is created. Therefore, structural requirement will be considered, in the situation when the mechanism is loaded (Fig. (1c)). In this case, the objective is to maximize the structural stiffness when force $-\mathbf{f}_B$ is applied to it and the point A is constrained in the direction of force $\mathbf{f}_A$. Due to this new constraint, a new stiffness matrix $\mathbf{K}_2$ is defined. The objective function of this part of the problem can be expressed as the objective function for the minimization of flexibility $-\mathbf{f}_B{}^T\mathbf{u}_B$, where $\mathbf{u}_B$ is the deflection field due to $-\mathbf{f}_B$. Substituting $-\mathbf{f}_B$ for $\mathbf{K}_2\mathbf{u}_B$, the objective function to be minimized can be posed as:

$$L_3 = (\mathbf{u}_B{}^T\mathbf{K}_2\mathbf{u}_B) \tag{4}$$

subjected to the constraints in Eq. (5)

The expression to be minimized is equal to twice the structure strain deformation energy due to $-\mathbf{f}_B$. Now, Eqs. (3) and (4) can be combined through a multi-criteria objective function. For the synthesis of single flexibility mechanisms it was used the ratio of the two objective functions (Frecker et al., 1996). Thus, the final optimization problem is given by:

$$\max_{A_i} F = \left[\frac{\mathbf{u}_A{}^T \mathbf{K}_1 \mathbf{v}_B}{\mathbf{u}_B{}^T \mathbf{K}_2 \mathbf{u}_B} \right]$$

subject to

$$\begin{aligned} &\mathbf{K}_1 \mathbf{u}_A = \mathbf{f}_A; \\ &\mathbf{K}_1 \mathbf{v}_B = \mathbf{f}_B; \\ &\mathbf{K}_2 \mathbf{u}_B = -\mathbf{f}_B; \\ &\sum_{i=1}^{M} A_i L_i \leq V_{max}; \\ &A_{min} \leq A_i \leq A_{max}, 1 \leq i \leq M \end{aligned} \tag{5}$$

In Eq. (5), M is the total number of trusses, V_{max} is the volume constraint, A_i are the cross-sectional areas of the elements, and A_{min} and A_{max} are the lateral constraints for the areas. The optimization problem is solved using the Sequencial Linear Programing method (SLP), as described in section 3. The SLP routine uses the sensitivities of the objective in relation to the design variables. The sensitivities are the partial derivatives in relation to each design variable. In this case:

$$\frac{\partial F}{\partial A_i} = \frac{\left(-\mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{v}_B \right) L_3 - L_2 \left(-\mathbf{u}_B{}^T \frac{\partial \mathbf{K}_2}{\partial A_i} \mathbf{u}_B \right)}{\left(L_3 \right)^2} \tag{6}$$

The development of this formula is made in the Appendix.

2.2. Design considering area penalization

In the synthesis of compliant mechanisms well-defined shape structures are searched. This means having trusses with cross-sectional areas near its lowest or highest value, to allow an easier interpretation. However, it is usual to obtain final topologies with intermediate cross-sectional areas. To reduce the number of elements having that characteristic, it is used a procedure called area penalization. It consists in substituting the element area A_i according to Eq. (7):

$$A_i = A_0 \cdot x_i{}^p \tag{7}$$

The parameter A_0 is the maximum element area, x_i is the design variable, and p is the area penalization factor. When x_i changes from its minimum value to 1, the area goes from A_{min} to A_{max}. The minimmum value has to be different from 0, to avoid singularities in the global stiffness matrix.

Using a p value different from 1, the stiffness no more depends linearly on the design variables. Particularly, for values of p higher than 1, the intermediate elements have lower stiffness than for p equal to 1. Moreover, until a certain value, which depends on p, design variable variation causes little changes in element stiffness. Therefore, penalization make costly the intermediate areas, due to volume increase without a significant change in stiffness. However, excessive values of p may cause numerical instabilities that result from the approximation of the continuous problem to a discrete one. The variations of the design variables can become excessively abrupt.

Due to penalization techniques, there are changes in the stiffness matrix, and, consequently, in relation to its sensitivities to the design variables.

2.3. Design considering multi-flexibility

Some compliant mechanisms may have more than one design requirement. For example, in the mechanism shown in Fig. (2), when load $\mathbf{F}_1$ is applied to point P_{11}, displacement Δ_1 is desired at point P_{12}. However, if load $\mathbf{F}_2$ is applied to P_{21}, the desired displacement is Δ_2, at point P_{22}.

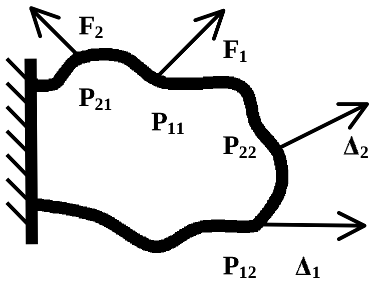


Figure 2. Multi-flexible mechanism.

To design this kind of mechanisms, two or more design criteria that incorporate the multiple flexibility cases are needed. All kinematic requirements will be satisfied if each flexibility case has mutual energy value at least higher than 0. As mutual energy gets higher, the structure becomes more flexible. Moreover, strain energy needs to be considered for each flexibility case. Therefore, mutual and strain energy are defined for each case, respectively as ${}^{i}L_2$ and ${}^{i}L_3$. Thus, the multi-flexibility objective function used in this work is given by Eq. (8) (Nishiwaki et al., 2001):

$$F_M = \left[-\frac{1}{\xi^L} \ln\left(\sum_{i=1}^{n} e^{-\xi^L\,{}^{i}L_2} \right) \Big/ \frac{1}{\xi^S} \ln\left(\sum_{i=1}^{n} e^{\xi^S\,{}^{i}L^3} \right) \right] \tag{8}$$

In this expression, ξ^L and ξ^S are constant values higher than 0, chosen by the user. Following, the expression of the sensitivities of the function F_M, obtained directly from derivation of Eq. (8), is described:

$$\frac{\partial F_M}{\partial x_i} = \frac{\left(\dfrac{\sum_{i=1}^{n} e^{-\xi^L\,{}^{i}L_2} \dfrac{\partial\,{}^{i}L_2}{\partial x_i}}{\sum_{i=1}^{n} e^{-\xi^L\,{}^{i}L_2}} \dfrac{\ln\left(\sum_{i=1}^{n} e^{\xi^S\,{}^{i}L_3} \right)}{\xi^S} + \dfrac{\sum_{i=1}^{n} e^{\xi^S\,{}^{i}L_3} \dfrac{\partial\,{}^{i}L_3}{\partial x_i}}{\sum_{i=1}^{n} e^{\xi^L\,{}^{i}L_3}} \dfrac{\ln\left(\sum_{i=1}^{n} e^{-\xi^L\,{}^{i}L_2} \right)}{\xi^L} \right)}{\left(\dfrac{\ln\left(\sum_{i=1}^{n} e^{\xi^S\,{}^{i}L_3} \right)}{\xi^S} \right)^2} \tag{9}$$

3. Numerical Implementation

The software for the synthesis of compliant mechanisms using Topology Optimization was developed using the C language. The user must specify size, shape and boundary conditions of the initial domain, the desired displacement, the applied loads and parameters such as: volume constraint, design variables lower and upper limits, initial guess for the pseudo-densities, material modulus of elasticity, maximum number of iterations and, in case of multi-flexibility, ξ^L and ξ^S. The initial design domain is meshed using enough number of truss elements to have a good approximation of the continuum. Considering area penalization, x_i are the design variables. Single flexibility mechanisms synthesis uses Eq. (5) as the objective function. To synthetize multi-flexibility mechanisms, objective function in Eq. (8) is used. Following, it is shown the flowchart of the optimization algorithm.

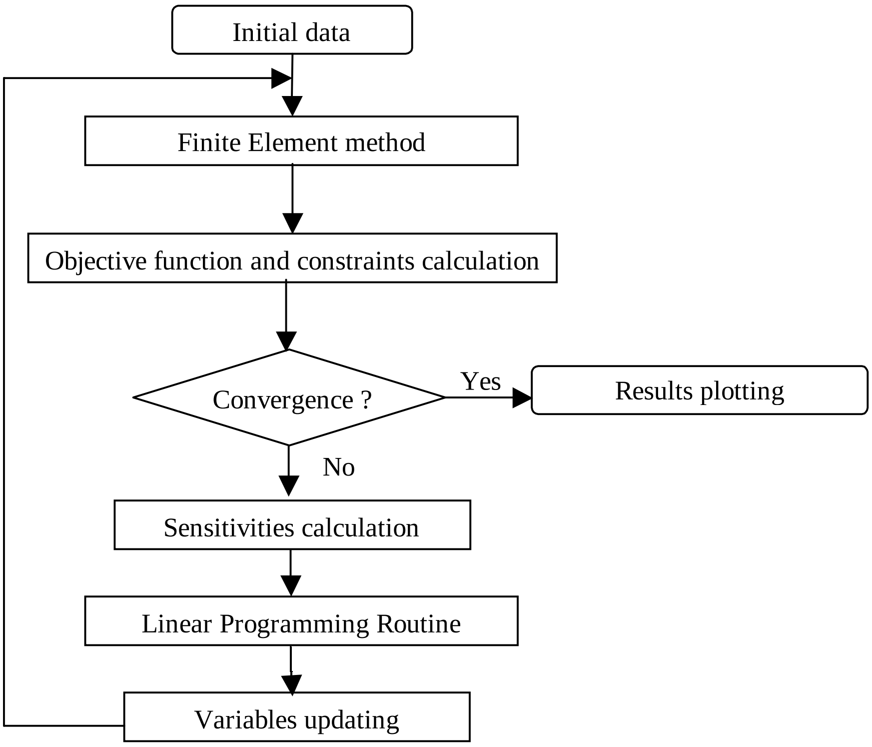


Figure 3. Flowchart of the iterative optimization process.

The optimization problem posed here is non-linear in relation to the design variables. To solve it, the software uses Sequencial Linear Programming (SLP). The SLP can deal with a great number of design variables and complex objective functions and constraints, needing only first derivatives of the objective function in relation to the design variables. Moreover, it can be easily computationally implemented. The SLP method consists in sequentially solving linear optimization problems. To obtain a linear problem, the objective function is linearized around the design variables actual value, using Taylor's Series of first order. The size of the analyzed interval is determined by the optimization pass. The routine used for solving linear optimization problem in this software is DSPLP, based on the KAMARKAR algorithm (Hanson and Hiebert, 1981).

4. Results

In this section, results obtained using the software developed in this work are presented and discussed. Section 4.1 presents a discussion about the use of area penalization in the design, using as example a compliant gripper. In section 4.2, a multi-flexibility problem is solved with the software. All results are simulated using finite element method through the commercial software ANSYS.

4.1. Design using area penalization

To exemplify the use of area penalization, a compliant gripper was synthetized, with different values of penalization. Due to its symmetry, and for computational time saving, only half of the structure is simulated. The design parameters are: V_{max}=15, x_{min}=0.01, x_{max}=1.0, $x_{initial}$=0.01, A_0=1.0, p=1.0. In Fig. (4), F represents the applied load and Δ the desired load.

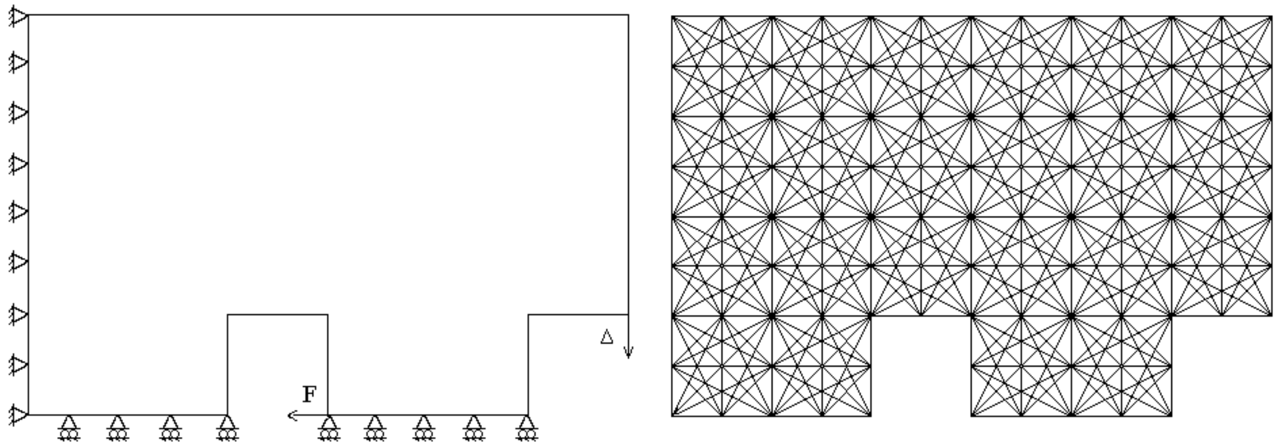
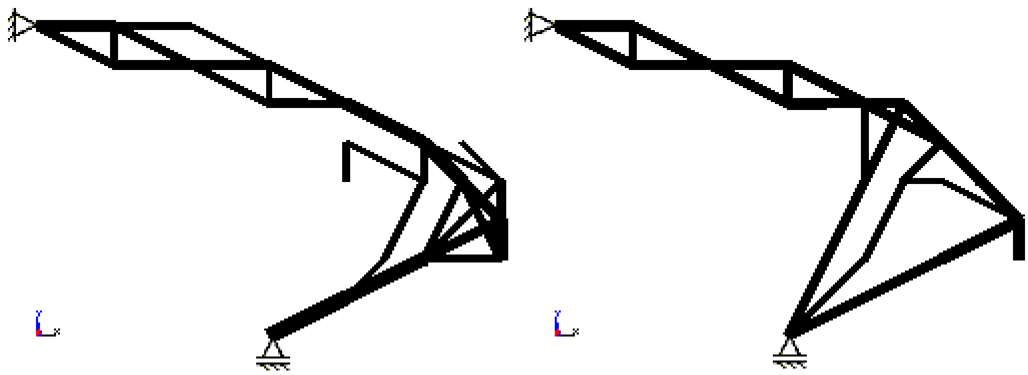


Figure 4. Initial design domain for a compliant gripper.

The first case uses penalization p=1.0. The second one uses p=1.8. The results are presented in Fig. (5):

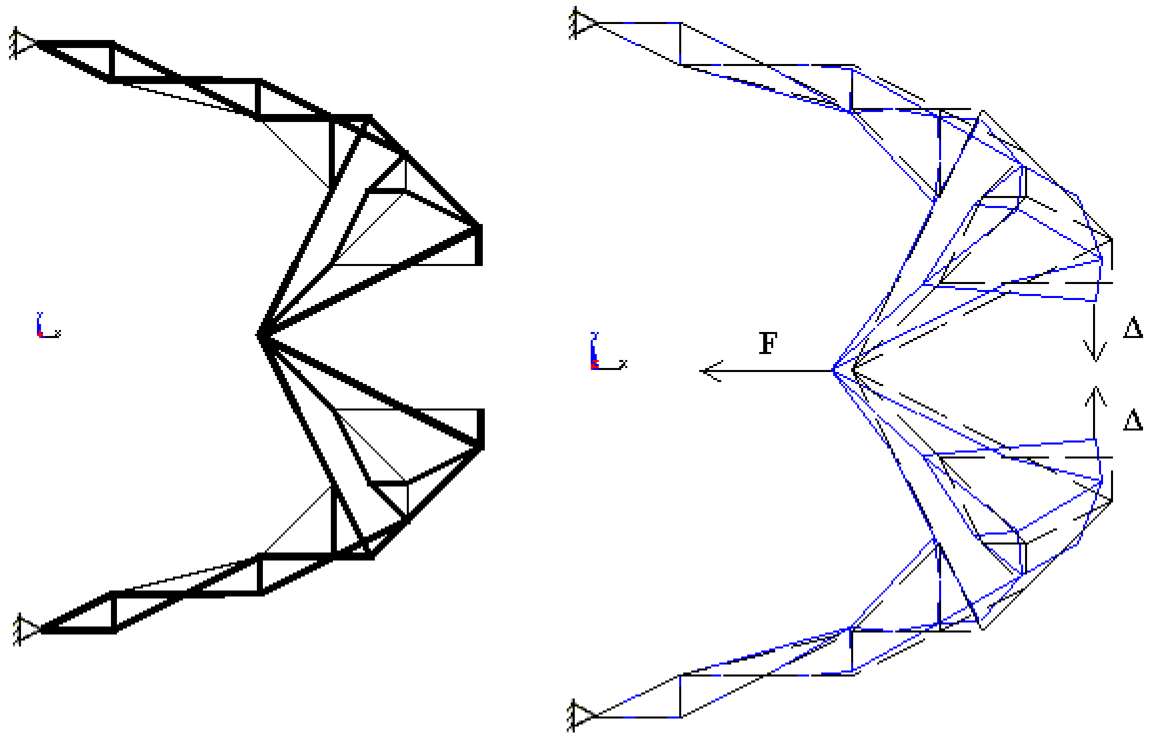


(a) (b)

Figure 5. Compliant gripper design: (a) p=1.0; (b) p=1.8.

Working with penalization value equal to 1.8 increases the cost of intermediate cross sectional areas. This occurs because they increase volume without changing significantly the objective function. Thus, solutions that use p higher than 1.0 tend to have bars with cross sectional areas close to its highest or lowest value. Therefore, the optimization process using area penalization generates better defined structures, as can be seen in Figs. (5a) and (5b). This makes interpretation of mechanisms easier.

It can be verified from Figs. (5a) that some bars make the structures to behave as rigid-body mechanisms. This occurs because the bars that would limit those movements have areas lower than $3*A_{min}$ and are not considered at the final topology. That fact causes problems in finite element simulation, once truss elements do not offer stiffness to rotations around nodes. It should be noted, though, that this problem is not observed in real compliant mechanisms. They are single pieces and, consequently, there is stiffness to bars rotation. To avoid problems during simulation, the topology is interpreted. The use of area penalization minimizes the need of interpretation. In Fig (6), topology and simulation of the complete gripper can be seen.



(a) (b)

Figure 6: (a) Interpretation of the obtained topology. (b) Finite element simulation of the entire gripper. In continuous lines, the deformed shape.

That result is similar to the result obtained by Nishiwaki et al. (1998). Volume and convergence curves are shown in Fig. (7).

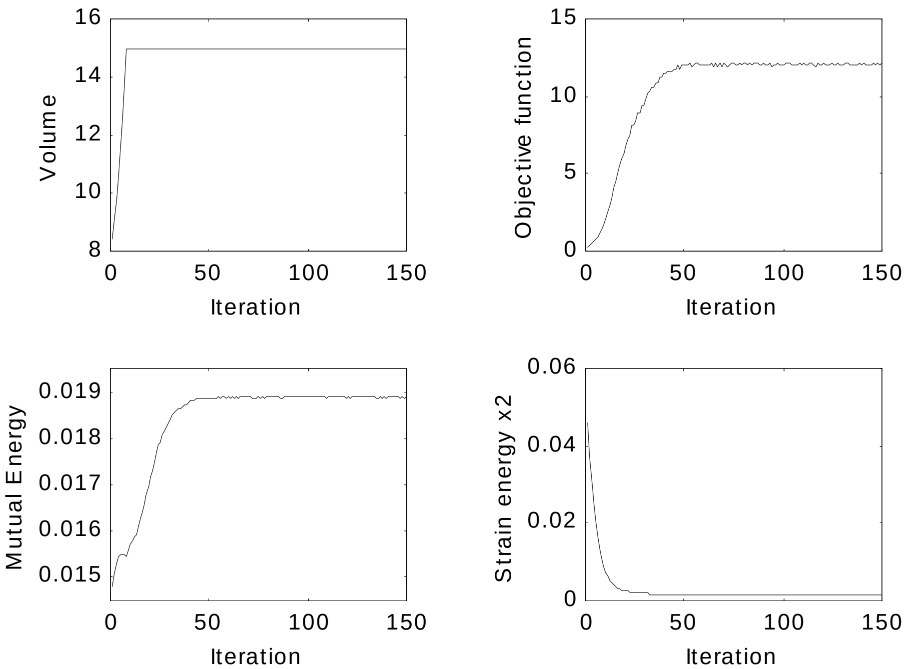


Figure 7. Volume and convergence curves.

Volume curve shows that in few iterations, the volume reaches the constraint. That occurs because, in this problem, strain energy is dominating the process. In convergence curves, it can be seen that mutual energy is maximized and strain energy is minimized. Consequently, the objective function is maximized.

4.2. Design considering multi-flexibility

The mechanism synthetized in this case may be loaded in two different ways. The first load is F_1, which must cause displacement Δ_1 (Fig. (8)). The second load is F_2, and the desired displacement is Δ_2. Design parameters are: $V_{max} = V_{initial}$; $x_{min}=0.001$; $x_{max}=1.0$; $x_{initial}=0.1$; $A_0=1.0$; $p=1.8$; $W=1$; $\xi_S = 1000$; $\xi_L=1000$.

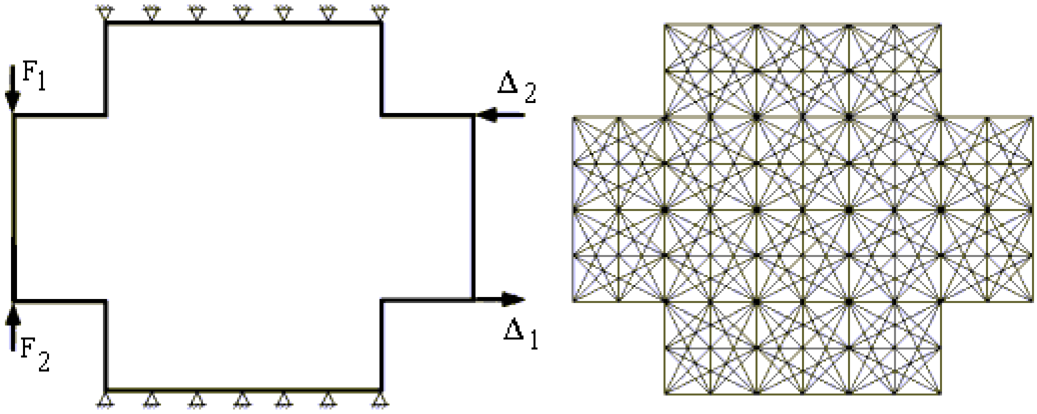


Figure 8. Initial design domain for multi-flexible mechanism.

The obtained topology is in Fig. (9):

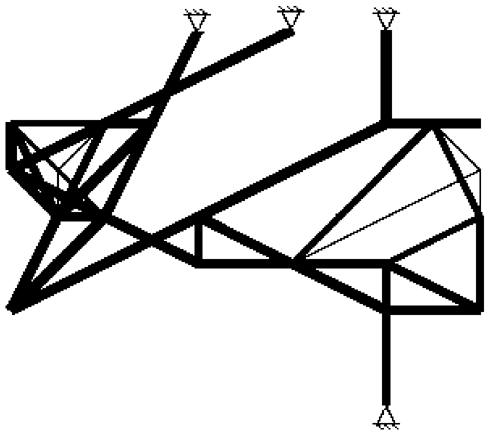
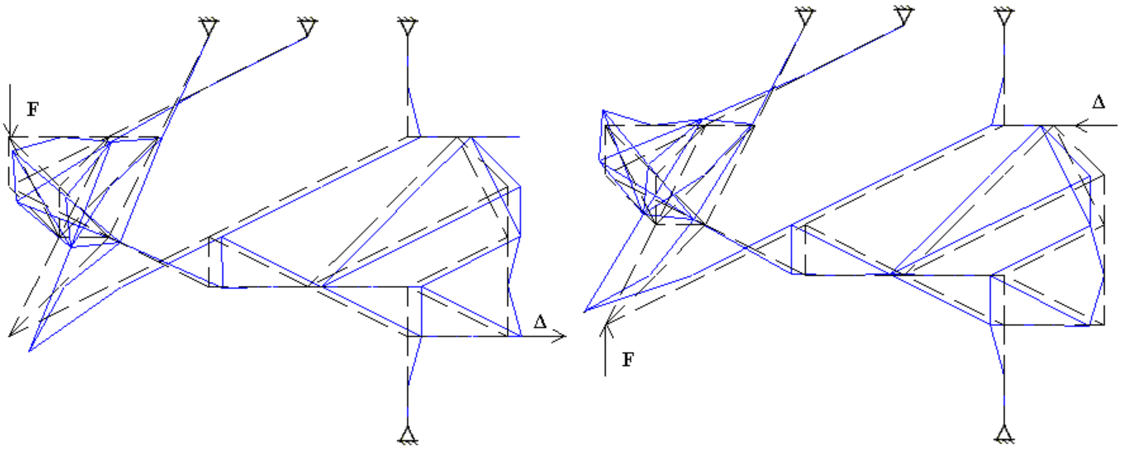


Figure 9. Final Topology.

Figure (10), shows the mechanism deformed shape under the two flexibility cases.



(a) (b)

Figure 10: (a) First flexibility case; (b) Second case.

Figure (10) shows that in both cases design objective is accomplished. The discrete solution presented here is similar to solutions presented for this problem using continuum domains by Nishiwaki et al. (2001).

5. Conclusion

A formulation for the synthesis of truss based compliant mechanisms using Topology Optimization and considering area penalization and multi-flexibility has been presented. The obtained topologies have been simulated using the Finite Element method. Simulations have showed that the mechanisms work according to design requirements. The examples confirm the efficiency of area penalization in avoiding intermediate cross sectional areas at the final topology. The results obtained are similar to those synthetized with continuum domains (Nishiwaki et al., 1998; Nishiwaki te al. 2001).

In the future, a formulation that substitutes truss elements for beam elements, that consider bending stiffness, is going to be implemented. The use of these elements may allow the model to be a more realistic approximation of real compliant mechanisms. Changes will also be made to incorporate tridimensional mechanisms synthesis. Another optimization method, based on the optimality criteria (Saxena et al., 2000) may be implemented to increase computational efficiency. In addition, prototypes of mechanisms generated by the software will be manufactured using MEMS micromanufacturing techniques.

6. Acknowledgements

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8. Appendix

Sensitivities of the objective function are calculated by the following procedure:

$$\frac{\partial F}{\partial A_i} = \frac{\partial\left(\frac{L_2}{L_3}\right)}{\partial A_i} = \frac{\frac{\partial(L_2)}{\partial A_i} L_3 - \frac{\partial(L_3)}{\partial A_i} L_2}{(L_3)^2} \tag{10}$$

However,

$$\frac{\partial(L_2)}{\partial A_i} = \frac{\partial \mathbf{u}_A{}^T}{\partial A_i} \mathbf{K}_1 \mathbf{v}_B + \mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{v}_B + \mathbf{u}_A{}^T \mathbf{K}_1 \frac{\partial \mathbf{v}_B}{\partial A_i} \tag{11}$$

By derivation of equilibrium equation $\mathbf{K}_1\, \mathbf{u}_A = \mathbf{f}_A$ in relation to the design variables:

$$\frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{u}_A + \mathbf{K}_1 \frac{\partial \mathbf{u}_A}{\partial A_i} = \frac{\partial \mathbf{f}_A}{\partial A_i} \tag{12}$$

Since $\mathbf{f}_A$ is constant in relation to A_i, $\partial\, \mathbf{f}_A / \partial A_i = 0$. Therefore:

$$\frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{u}_A + \mathbf{K}_1 \frac{\partial \mathbf{u}_A}{\partial A_i} = 0 \Rightarrow \frac{\partial \mathbf{u}_A}{\partial A_i} = -\mathbf{K}_1{}^{-1} \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{u}_A = 0 \Rightarrow \frac{\partial \mathbf{u}_A{}^T}{\partial A_i} = -\mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \left(\mathbf{K}_1{}^{-1}\right)^T = \\ = \mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{K}_1{}^{-1} \tag{13}$$

As long as $\mathbf{K}_1$ is symmetric, $\mathbf{K}_1{}^T = \mathbf{K}_1$ and $(\mathbf{K}_1{}^{-1})^T = \mathbf{K}_1{}^{-1}$. Derivation of equation $\mathbf{K}_1\, \mathbf{v}_{B\,=} = \mathbf{f}_B$:

$$\frac{\partial \mathbf{v}_B{}^T}{\partial A_i} = -\mathbf{K}_1{}^{-1} \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{v}_B \tag{14}$$

Substituting Eqs. (13) and (14) in (11) and simplifying:

$$\frac{\partial(L_2)}{\partial A_i} = -\mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{v}_B \tag{15}$$

For the derivative at the second part of the numerator of Eq. (10), the same procedure can be followed. Therefore:

$$\frac{\partial(L_3)}{\partial A_i} = -\mathbf{u}_\mathbf{B}{}^T \frac{\partial \mathbf{K}_2}{\partial A_i} \mathbf{u}_\mathbf{B} \tag{16}$$

From Eqs. (15) e (16) in (10), comes:

$$\frac{\partial F}{\partial A_i} = \frac{\left(-\mathbf{u}_A{}^T \frac{\partial \mathbf{K}_1}{\partial A_i} \mathbf{v}_B\right) L_3 - L_2 \left(-\mathbf{u}_B{}^T \frac{\partial \mathbf{K}_2}{\partial A_i} \mathbf{u}_B\right)}{(L_3)^2}$$