THERMAL POST-BUCKLING ANALYSIS OF SLENDER ELASTIC RODS WITH NON-MOVABLE DOUBLE-HINGED ENDS

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Abstract. *Mathematical formulation and analytical solution are presented for the thermal post-buckling behaviour of slender elastic rods constrained by double-hinged non-movable boundary conditions. The material is assumed linear elastic, homogeneous and isotropic and its thermal strain-temperature relationship is non-linear. Large displacements are considered hence the formulation is geometrically non-linear. The governing equations are derived from geometrical compatibility, equilibrium of forces and moments, constitutive equations and strain-displacement relation, yielding a set of six first-order non-linear ordinary differential equations with boundary conditions specified at both ends, which constitutes a complex boundary value problem. A closed-form analytical solution found via complete elliptic integral is derived from the governing equations defining the shape of the post-buckled rod (elastica). The results are presented in non-dimensional graphs for a range of temperature gradients and different values of slenderness ratios. The consideration of slender rods allows extending the formulation for pipelines. The phenomenon of thermal buckling in pipelines, through analytic and numeric models, including geometric non-linearity is then studied.*

Keywords. *Post-buckling, thermal buckling, elastic rods*

1. Introduction

The problem of elastic stability of rods subjected to mechanical and thermal compressive loads has been well studied since Bernoulli, Euler and Lagrange investigated the classical problem of the *elastica*, i.e., the equilibrium configurations of inextensible rods under axial compression. Love's (1944) seminal textbook on theory of mathematical elasticity has been extensively used in many fields of applied mechanics, establishing the basis for most research on the equilibrium of elastic rods. Some papers were published on buckling and post-buckling behaviour obtaining solutions for the differential equation that governs the elastic line of an initially straight slender rod (the *elastica* problem) subjected to different of compressive loads and boundary conditions, Theocaris and Panayotounakos (1982), Stemple (1990), Wang (1997), Filipich and Rosales (2000) and Vaz and Silva (2002).

The problem of elastic stability of rods subjected to thermal loads and mechanical compressive loads are substantially different and in fact not as many articles have been published regarding thermal buckling of rods. El Naschie (1976) considered the thermal stability of an extensional rod. Buckling and post-buckling behaviour in the sense of Koiter were treated within the framework of the general branching theory of discrete systems. In his paper Jekot (1996) investigated the thermal post-buckling of a beam made of physically non-linear thermoelastic material. The range for safe buckling temperature was determined, and some comparisons between the non-linear and linear post-buckling behaviour were discussed. However, the geometric non-linearity due to the central axis curvature was not considered and a simplified form of the non-linear axial strain was used. Coffin and Bloom (1999) developed an elliptic integral solution for the post-buckling response of a linear-elastic and hygrothermal beam fully restrained against axial expansion. They assumed linear thermal strain-temperature relationship and solved the set of differential equations for the undeformed configuration, hence two coupled integral elliptic equations needed to be simultaneously solved. Based on the exact non-linear geometric theory for extensible rods and using a shooting method, a computational analysis for the thermal post-buckling behaviour of rods with axially non-movable pinned-pinned ends as well as fixed-fixed ends was proposed by Li and Cheng (2000). More recently, Li et all (2002) presented a mathematical model for the post-buckling of an elastic rod with pinned-fixed ends when a quasi-static increasing temperature is applied. Using the shooting method in conjunction with the concept of analytical continuation, the non-linear boundary value problem consisting of ordinary differential equations was numerically solved. The results showed that the critical buckling temperature and the post-buckled rod configuration were sensitively influenced by the slenderness ratio. Cisternas and Holmes (2002) included thermal expansion effects in the extensible rod theory, focusing his study on the bifurcations of the resulting equilibrium equations under both traction and displacement boundary conditions and determined sub-critical and supercritical pitchfork bifurcations.

Such a very narrow relationship between the thermal buckling of slender components - such as railroad tracks, concrete road pavements, optical fibers, satellite tethers or subsea and buried pipelines - and the buckling of rods has long been recognized. It is therefore of practical design interest to employ simplified analysis. The structural behaviour of pipelines was appraised by Hobbs (1984) in a classical paper that has been extensively accepted for industrial design. Similar studies were presented by Taylor and Gan (1996), Ju and Kyriakides (1988), and Pedersen e Michelsen (1988), but they considered the effect of initial imperfection. The recent increase of the necessity of high temperature flowlines and the lack of publications about the subject unleashed the interest on the study of this phenomenon. Several papers that describe the structural behaviour of pipelines subjected to action of thermal loading are of summa importance to start of the study.

This paper investigates the post-buckling response of an initially straight slender rod made of linear elastic material whose strain-temperature relationship is non-linear. A temperature gradient is assumed uniform along the rod and expansion is prevented by double-hinged non-movable ends. The solution is obtained by uncoupled elliptic integrals, which are derived from the governing equations in the deformed configuration, hence completely defining the shape of the rod (*elastica*). This study may be qualitatively expanded to pipes and other slender structures subjected to thermal loads.

2. Mathematical Formulation

The governing equations are derived from the geometrical compatibility, equilibrium of forces and moments, constitutive equations and strain-displacement relation, which are presented next. Consider a deflected slender rod with non-movable double-hinged ends in its initial and buckled configurations, as shown in Fig. 1.

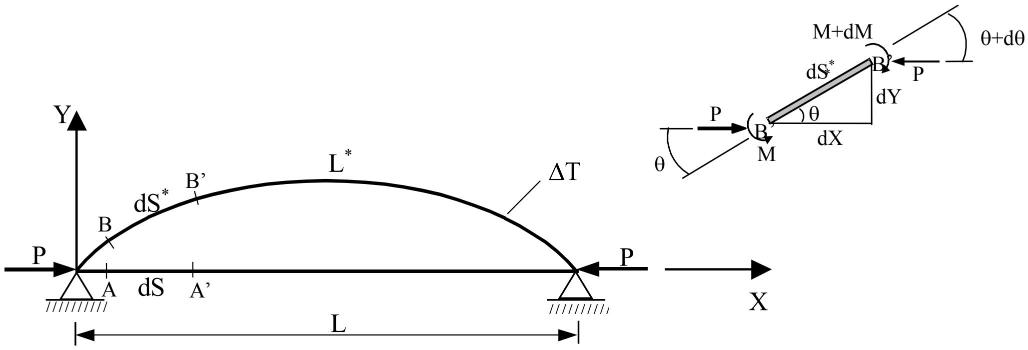

Figure 1. Schematic of a Deflected Elastic Rod under Uniform Thermal Load.

Where the pair (X, Y) constitutes the Cartesian coordinate system, ΔT is the uniform temperature gradient, P is the compressive load arising from the expansion constraint, L is the initial rod length and L^* is the deformed rod length.

2.1. Geometrical compatibility

Applying trigonometrical relations to the infinitesimal deformed element of rod dS^* (see Fig. 1) yields:

$$\frac{dX}{dS^*} = \cos\theta \tag{1a}$$

$$\frac{dY}{dS^*} = \sin\theta \tag{1b}$$

Where S^* is the deformed arc-length ($0 \le S^* \le L^*$) and θ is the angle between the tangent and the X-axis.

2.2. Equilibrium of forces and moments

A schematic of the internal forces and moments in the deformed infinitesimal element of the rod is presented in Fig. 1. The equilibrium of moments at B' ($\sum M_{B'} = 0$), for instance, yields:

$$\frac{dM}{dS^*} = P\sin\theta \tag{2}$$

Where M is the bending moment.

Equilibrium of forces in the X-direction $(\sum F_X = 0)$ results in a constant compressive load P along the rod. Therefore:

$$\frac{dP}{dS^*} = 0 \tag{3}$$

Note that for double-hinged ends there is no component of reaction forces in the Y-axis.

2.3. Constitutive relations

Assuming linear elastic, homogeneous and isotropic materials (constitutive relations given by Hooke`s Law), and considering the state of pure bending results in:

$$M = -EI\,\mathrm{K} \tag{4}$$

Where E is the modulus of Young and I is the cross-sectional inertia. The curvature K is given by:

$$\frac{d\theta}{dS^*} = \mathrm{K} \tag{5}$$

Therefore, substituting Eq. (4) into (2) results:

$$\frac{d\mathrm{K}}{dS^*} = -\frac{P}{EI}\sin\theta \tag{6}$$

2.4. Strain-displacement relation

For an infinitesimal element the specific linear strain ε (or relative elongation) is defined as being the relation between the elongations suffered by the element, when passing to the deformed configuration, and its initial length:

$$\frac{dS}{dS^*} = \frac{1}{1+\varepsilon} \tag{7}$$

When a slender rod is subjected to a temperature gradient ΔT it tends to expand and, consequently, a compressive load P appears if movement of the ends is restricted. Hence, the total strain is given by the addition of the thermal strain and the strain due to the compressive load ($\varepsilon = \varepsilon_T + \varepsilon_c$):

$$\varepsilon = \alpha\,\Delta T + \frac{\overline{L}}{E}\alpha^2\Delta T^2 - \frac{P}{EA}\cos\theta \tag{8}$$

Where α is the thermal expansion coefficient, A is the cross-sectional area,

$$\overline{L} = l(1-2\nu) - 2m(\nu^2 - 1) + n\nu^2$$

l, m, n are Murnaghan's constants and ν is the Poisson's ratio. The first two terms on the right hand side of Eq. (8) define the thermal strain for materials whose strain-temperature dependence is non-linear, as proposed by Smith et all (1966). Note that for metals $\overline{L}$ may only assume negative values. Substituting Eq. (8) into (7), and rearranging yields:

$$\frac{dS}{dS^*} = \frac{1}{1+\left(\alpha\Delta T + \frac{\overline{L}}{E}\alpha^2\Delta T^2 - \frac{P}{EA}\cos\theta\right)} \tag{9}$$

2.5. The governing equations

In summary, the governing equations for slender elastic rods with double-hinged non-movable ends subjected to uniform thermal induced load are given by Eqs. (1a), (1b), (3), (5), (6) and (9).

Furthermore, variables may be made non-dimensional by using the following relations: $L^* = l^* L$ $S = s\,L$, $X = x\,L$, $Y = y\,L$, $S^* = s^* L$, $\mathrm{K} = \kappa / L$, $\lambda^2 = L^2 A / I$, $P = pEI / L^2$, $\gamma = \overline{L} / E$, and $\Delta T = \Delta t / \lambda^2 \alpha$. Consequently, the governing equations are rewritten as:

$$\frac{dx}{ds^*} = \cos\theta \tag{10a}$$

$$\frac{dy}{ds^*} = \sin\theta \tag{10b}$$

$$\frac{d\theta}{ds^*} = \kappa \tag{10c}$$

$$\frac{ds}{ds^*} = \frac{1}{(1+\varepsilon)} \tag{10d}$$

$$\frac{d\kappa}{ds^*} = -p\sin\theta \tag{10e}$$

$$\frac{dp}{ds^*} = 0 \tag{10f}$$

Where:

$$\varepsilon = \gamma\frac{\Delta t^2}{\lambda^4} + \frac{\Delta t}{\lambda^2} - \frac{p}{\lambda^2}\cos\theta \tag{10g}$$

is the central line strain and the constant λ is the rod slenderness ratio and γ is the non-linear thermal strain coefficient. And the boundary conditions are $x(0) = y(0) = \kappa(0) = x(l^*) - 1 = y(l^*) = \kappa(l^*) = 0$.

2.6. Determination of the critical buckling temperature

The calculation of buckling loads follows straightforward approximation of the moment equilibrium equation by assuming small displacements, i.e., $\cos\theta \cong 1$ and $sin\theta \cong \theta$. So, the governing equation may be reduced to:

$$\frac{d^4y}{dx^4} + p\frac{d^2y}{dx^2} = 0 \tag{11}$$

General solution for the homogeneous differential Eq. (11) with constant coefficients is quickly found:

$$y(x) = C_1\sin\left(\sqrt{p}\,x\right) + C_2\cos\left(\sqrt{p}\,x\right) + C_3x + C_4 \tag{12}$$

Application of boundary conditions for rods with double-hinged ends yields $C_2 = C_3 = C_4 = C_1\sin(\sqrt{p}) = 0$, and to avoid trivial solution C_1 must be different from zero, which can be satisfied if $\sin(\sqrt{p}) = 0$ and $\sqrt{p} = n\pi$, where n is a positive integer. The smallest eigenvalue in this case corresponds to $n = 1$, i.e., the first buckling mode corresponds to:

$$p_c = \pi^2 \tag{13}$$

Subjected to a uniform temperature increase, the rod tends to expand, but until it reaches the critical buckling load, its strain is zero $(\varepsilon = 0)$, hence:

$$\gamma\frac{\Delta t^2}{\lambda^2} + \Delta t - p = 0 \tag{14}$$

Equation (13) can be substituted in Eq. (14) to find the critical buckling temperature:

$$\Delta t_c = \frac{\lambda^2}{2\gamma}\left(-1 + \sqrt{1 + 4\pi^2\gamma/\lambda^2}\right) \tag{15}$$

Equation (15) indicates that two parameters control the critical buckling temperature, the rod slenderness ratio λ and the non-linear thermal strain coefficient γ .

3. Analytical Post-Buckling Solution

A closed-form analytical solution for the thermal buckling of slender elastic rod is developed next for non-linear strain-temperature relationship via complete elliptic integral derived from the governing equations in the deformed configuration (Timoshenko and Gere (1961)). Instead of deflection y it is more convenient to work with the slope angle θ (Bažant and Cedolin (1991)), so the non-dimensional differential Eqs. (10c) and (10e) yield:

$$\frac{d^2\theta}{ds^{*2}} = -p\sin\theta \tag{16}$$

It so happens that this non-linear equation can be easily solved employing elliptic integrals. The solution was given in 1859 by Kirchhoff who noticed that it is mathematically identical to the equation that describes large pendulum oscillations, which had been earlier solved by Lagrange (a kinetic analogy of columns; Love (1944)). Integrating the Eq. (16) and applying the boundary conditions at the ends of the rod yield:

$$\frac{1}{2}\left(\frac{d\theta}{ds^*}\right)^2 = p\cos\theta + C \tag{17}$$

Where $C = -p\cos\beta$ and $\theta(0) = -\theta(l^*) = \beta$. Hence:

$$\frac{d\theta}{ds^*} = -\sqrt{2p(\cos\theta - \cos\beta)} \tag{18}$$

Recurring to familiar trigonometric identities to rewrite the Eq. (18), separating and changing variables ($\sin(\theta/2) = c\sin\phi$, where $c = \sin(\beta/2)$) and after some algebraic manipulation followed by an integration yield the deformed rod length:

$$l^* = \frac{2}{\sqrt{p}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}} \tag{19}$$

The x and y coordinates for the slender rod deflected configuration may be obtained from the non-dimensional Eqs. (10a) and (10b):

$$x = \frac{1}{\sqrt{p}}\int_{\phi_0}^{\pi/2}\frac{1-2c^2\sin^2\phi}{\sqrt{1-c^2\sin^2\phi}}\,d\phi \tag{20a}$$

$$y = \frac{2c}{\sqrt{p}}\cos\phi_0 \tag{20b}$$

And $-\pi/2 \le \phi_0 \le \pi/2$. Since $\kappa = -p\,y$ the rod curvature κ at the deformed configuration may now be readily obtained:

$$\kappa = -\frac{2pc}{\sqrt{p}}\cos\phi_0 \tag{21}$$

Symmetry implies that the point of maximum displacement occurs for $x(l^*/2) = 1/2$, so one may calculate p for this condition as a simple application of Eq. (20a):

$$p = \left[2\int_0^{\pi/2}\frac{1-2c^2\sin^2\phi}{\sqrt{1-c^2\sin^2\phi}}\,d\phi\right]^2 \tag{22}$$

Therefore for each deformed configuration (which is related to a temperature gradient), i.e., for a given slope β, $c = \sin(\beta/2)$ is calculated, and consequently p from Eq. (22). Finally, it is possible to find the coordinates (x, y) and curvature κ along the rod from Eqs. (20a), (20b) and (21).

The temperature gradient associated with the deformed configuration may be obtained considering the Eq. (10g) with some algebraic manipulation followed by a grouping of some terms concentrated in the parameter A. Thus:

$$\Delta t = \frac{\lambda^2}{2\gamma}\left(-1+\sqrt{1+4A\gamma/\lambda^2}\right) \tag{23}$$

Where: $A = \lambda^2\left(l^* - 1\right) + 2\sqrt{p}\int_0^{\pi/2} \frac{\left(1-2c^2\sin^2\phi\right)^2}{\left(1-c^2\sin^2\phi\right)^{1/2}}\,d\phi$.

This expression may be readily evaluated once p and l^* are known.

4. Analysis of Results

The closed-form solution is implemented through a computational program developed in the mathematical software Mathcad 2001 (2000) and a parametric study is carried out with the purpose of analyzing the results. The thermal strain-temperature relationship is considered non-linear and the most significant results regarding the phenomenon of thermal post-buckling of rods are presented for typical values of slenderness ratio: deformed configuration, maximum deflection, maximum inclination angle, maximum curvature, compressive load and total deformation. The results are presented for $\gamma = 0$ and $\gamma = -5$ (metallic materials, see Smith et all (1966)), which respectively correspond to materials with linear and non-linear strain-temperature relationships. The coefficient $\gamma = -5$ is relatively large for steels, but it is assumed to emphasize non-linear effects.

The physical and geometrical properties should be carefully selected to ensure practical and real meaning to the analysis, as well as avoiding nonconformity with the assumptions from the mathematical formulation. High temperatures and strains above 2% should not be considered. Furthermore, the parametric study was conducted for rod slenderness ratios $\lambda =$ 50, 100, 150 and 200.

The rod is initially straight, deprived of initial imperfections or initial temperature induced strain. The temperature is evenly increased and the total rod deformation ε is monitored with the objective of establishing the maximum allowable temperature for each slenderness ratio. The temperatures at which the rod deforms 1% and 2%, considering linear ($\gamma = 0$) and non-linear ($\gamma = -5$) strain-temperature relationships, are presented in Tab. 1.

Table 1. Temperature Gradients for Linear and Non-Linear Analyses.

	Δt ($\gamma = 0$ / $\gamma = -5$)	
λ	ε=1%	ε=2%
50	34.6 / 37.4	59.4 / 68.9
100	109.6 / 116.4	209.4 / 237.6
150	234.6 / 248.3	459.4 / 519.3
200	409.6 / 433.1	809.4 / 913.8

Once the critical buckling load is reached by Eqs. (13) and (15), and temperature is progressively increased, the compressive force arising in the boundaries falls considerably, as Eq. (22). This is clearly observable in the Fig. 2, where the variation of the compressive load (p) versus the temperature gradient for rods subjected to linear ($\gamma = 0$) and non-linear ($\gamma = -5$) thermal strain-temperature relationship is presented.

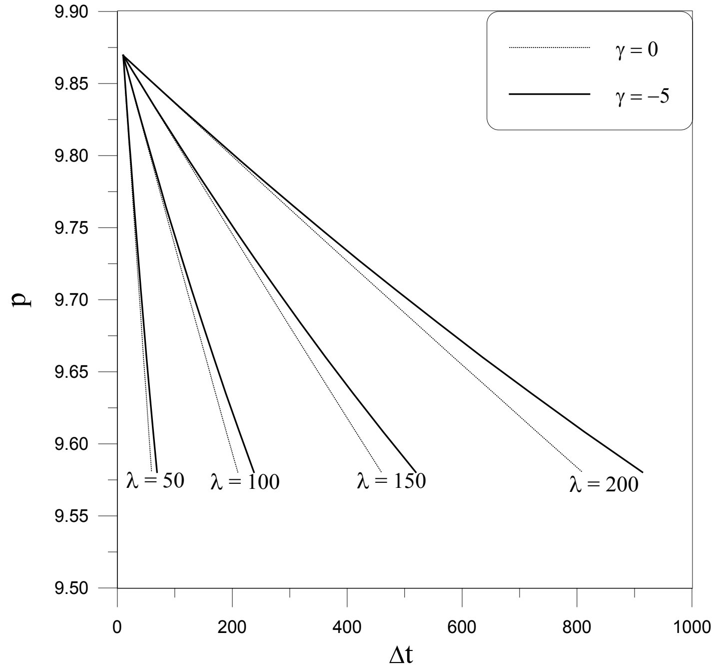

Figure 2. Compressive Load as a Function of the Temperature.

Figure 3 displays the maximum total deformation (ε_{total}) which takes place at the ends of the rod. The deformation begins when the critical buckling temperature is reached, and as expected it rises with temperature, according to Eq. (10g).

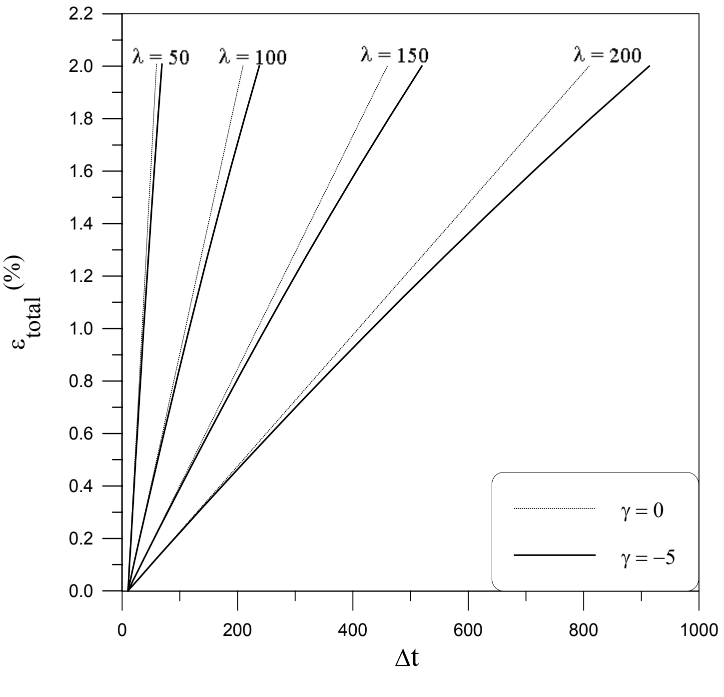

Figure 3. Total Deformation as a Function of the Temperature.

Figures 4 to 6 respectively present the results for maximum deflection ($y_{\max}$), maximum inclination angle ($\theta_{\max}$) and maximum curvature ($\kappa_{\max}$) in the rod, originating from Eqs. (20b), (16) and (21) - which respectively occurs at the middle, extremes and middle of the rod length - as a function of the temperature.

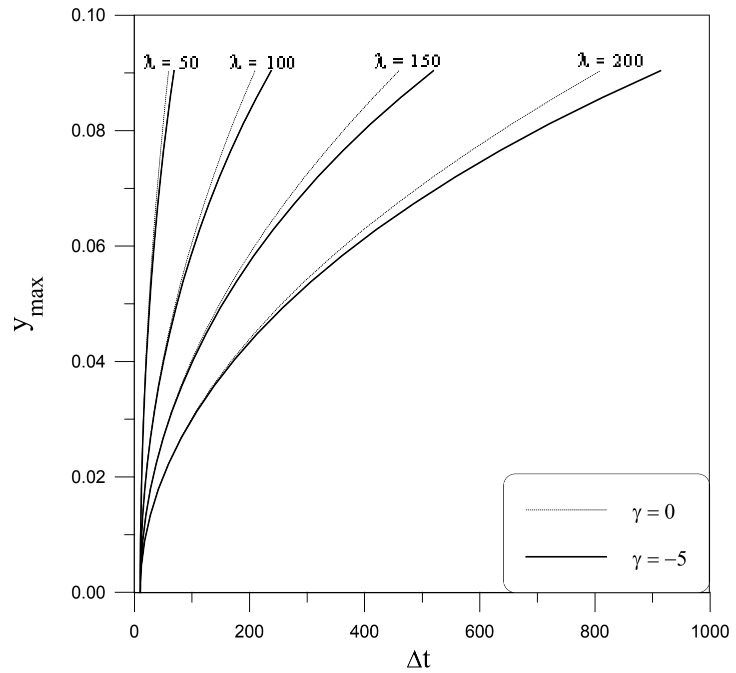

Figure 4. Maximum Deflection as a Function of the Temperature.

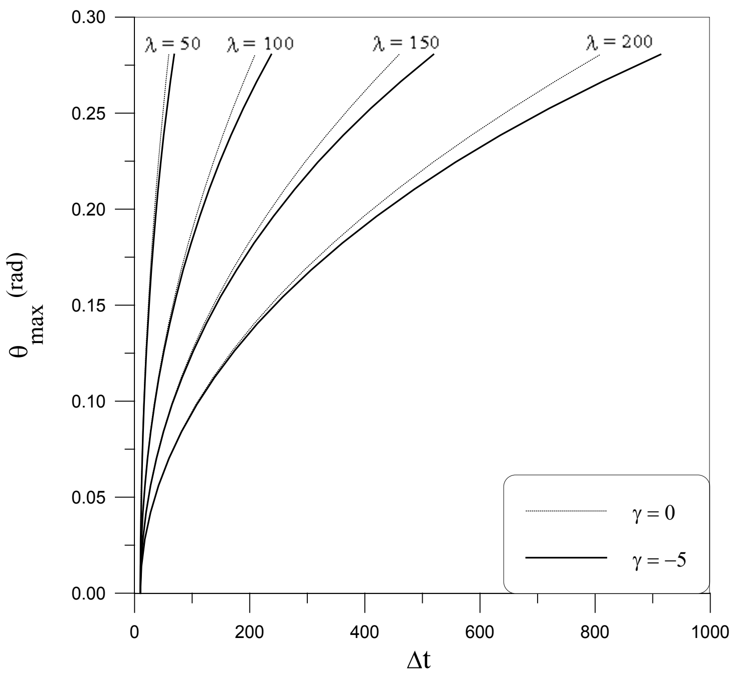

Figure 5. Maximum Angle as a Function of the Temperature.

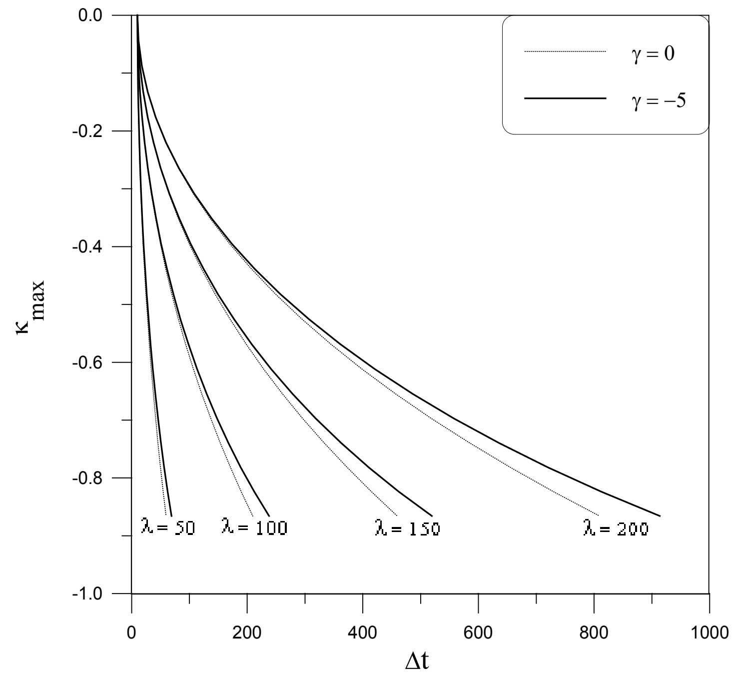

Figure 6. Maximum Curvature as a Function of the Temperature.

Figure 7 presents the deformed configurations as a function of temperature gradients (Eqs. (20a) and (20b)) for slenderness ratios $\lambda =$ 50, 100, 150 and 200. The configurations are plotted for temperatures at which the rod deforms 0.18, 0.84 and 2%. For smaller slenderness ratios, linear ($\gamma = 0$) and non-linear ($\gamma = -5$) analyses differ more significantly.

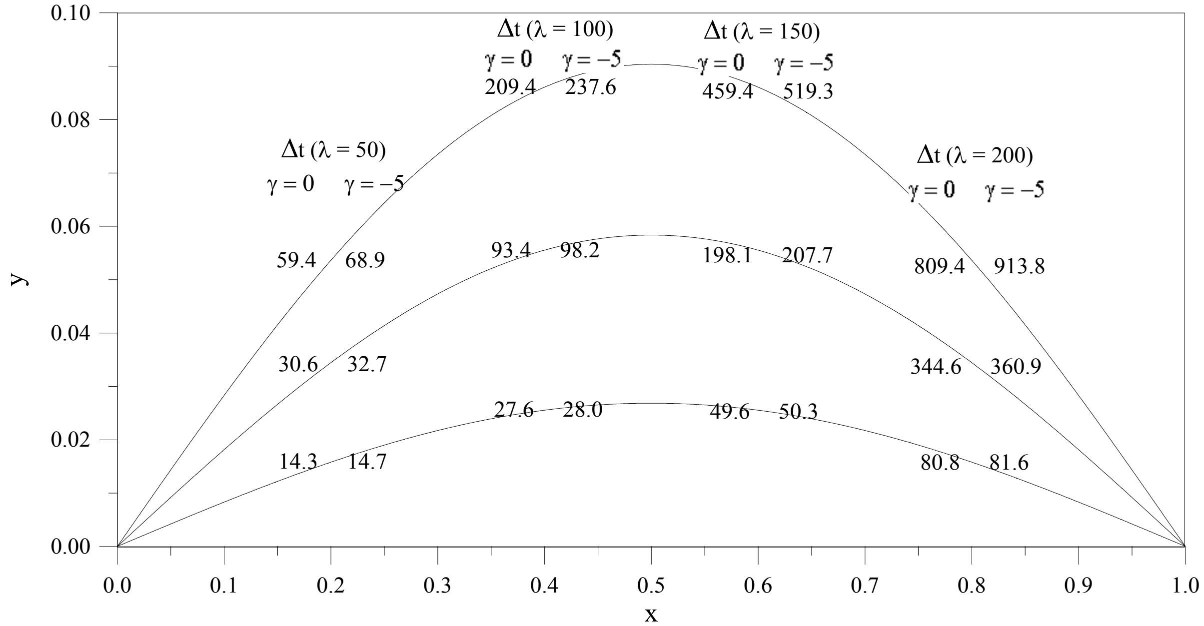

Figure 7. Deflected Configurations for a Range of Slenderness Ratio and Temperature Gradients.

5. Conclusions

The formulation and solution for a non-linear post-buckling analysis of slender elastic rods subjected to uniform temperature gradient are successfully accomplished in this article. The rod is assumed constrained by double-hinged non-movable boundary conditions and analysis is carried out for materials whose strain-temperature relationship is linear and non-linear. The governing equations are written in non-dimensional form and it is seen that two parameters control the solution: the slenderness ratio and the non-linear thermal strain coefficient. The critical buckling temperature is also calculated. The closed-form analytical solution allows analysis for rod expansion and its deformed geometric configuration as the temperature gradient progressively increases. A comparison study is carried out and results are obtained for a range of temperature as a function of the slenderness ratios and non-linear thermal strain coefficients.

This study offers a solid mathematical formulation and forms a base for future research with the objective of numerically modeling the pipeline post-buckling behaviour.

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